GENERAL EXACT INVERSE SOLUTION TO STEADY TWO-DIMENSIONAL HEAT CONDUCTION WITH HEAT GENERATION IN A PLANE WALL

حل عام عكسي مضبوط لأنتقال حراري مستقر ثنائي الابعاد مع توليد حراري في حائط مستوى

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ملخص :

في هذا البحث تم ايجاد حل عام مضبوط لنوع معين من المسائل المعكوسة لأنتقال حراري مستقر ثنائي الابعاد في جسم مسطح تتولد حرارة بداخله بانتظام. هذا الحل العام تم وضعه في صورة صريحة لحساب درجات الحرارة والفيض الحراري. هذا وقد تم اثبات صحة هذا الحل العام بتطبيقه علي مسائل لها حلول معلومة ، كما ان تاثير التوليد الحراري الداخلي تم تمثيله تمثيلاً عاماً دقيقاً في الحل العام، كما يعتبر هذا الحل ذو اهمية عملية ايضا.

ABSTRACT

A certain type of the inverse problems of steady heat conduction with heat generation in a two-dimensional plane wall has been analyzed. The analysis results in a general exact solution formulated in explicit expressions for temperature and heat flux calculation. Validity of the proposed solution have been proved by test problems having known exact solutions. The effect of heat generation in the wall is accurately modeled in the general solution.

1. INTRODUCTION

In many steady-state, heat transfer investigations using a flat plate as the test section, some technical difficulties may arise, if it is desired to measure temperature at more than one boundary surface of the plate. Further, the presence of a thermocouple at the effective heat transfer surface of the tested plate may affect the heat transfer modes close to the temperature sensor. Moreover, the conductive heat transfer in the plate body may be significant in two directions due to effect of the investigated phenomenon or/and the distribution feature of the applied heat flux. In such a practical situation, it may be necessary to solve the two-dimensional heat conduction problem in the plate wall in order to predict accurately the temperatures and heat flux at the effective heat transfer surface using only some corresponding measurements at the opposite plate surface. This problem is not a classical boundary value problem characterizing by two boundary conditions in every direction, but it is identified as inverse problem in heat conduction literatures. Beck [1] has classified the inverse problems to steady, and transient problems.

In the last three decades or so, there has been considerable interest in the solution of the transient inverse problems. Most of those studies have been performed numerically (e g, [2-4]), while the analytic solutions (e g, [6,7]) are still scarce and restricted to the one-dimensional case due to the difficulty of a multi-dimensional solution [5].

In 1964, Burggraf [6] derived an exact analytic solution; in a series form, for the inverse problem of transient, one-dimensional, heat conduction. This solution reads :

$$T(x,\tau) = \sum_{n=0}^{\infty} \frac{(x)^{2n}}{(2n)!\,\alpha^{n}} \frac{d^{n}T_{o}}{d\tau^{n}} - \frac{1}{k} \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{(2n+1)!\,\alpha^{n}} \frac{d^{n}q_{o}}{d\tau^{n}} \qquad (1)$$

where α is the thermal diffusivity, T_o $(=T(0,\tau))$ and q_o $(=q_x(0,\tau))$ are known boundary conditions which should be continuous and differentiable functions of the time variable τ.

Recently, general exact solution for a certain type of the inverse problems of steady, two-dimensional heat conduction has been derived for a planar wall; in (x,y)-cartesian coordinates [9]. This steady solution is somewhat similar to the transient solution of Burggraf (cf Eq. (1)). The analogy between the two solutions is in that the y variable in the steady solution simulates the role of the time variable in the transient solution.

It is important to note that the above mentioned steady solution [9] is limited to the case of no heat generation in the wall. However, if this effect could be modeled in the solution, the method will become of more theoretical as well practical interest. Therefore, the present paper is concerned with estimating two-dimensional, steady-state, temperature distribution in a planar wall involving heat generation, by utilizing the distributions of the temperature and the exterior heat flux; both prescribed at the same boundary surface as functions of the spatial coordinate along the surface. Validity of the present solution is proved.

2. ANALYSIS

Consider a rectangular plate with internal heat generation which is confined by the region: $0 \le x \le L$, $0 \le y \le \iota$ (cf Fig. 1). The temperature and the exterior heat flux at the boundary surface (0,y) are known continuous and differentiable functions of the variable y. Our main objective is to obtain the steady (x,y)-field of temperature using only these two boundary conditions.

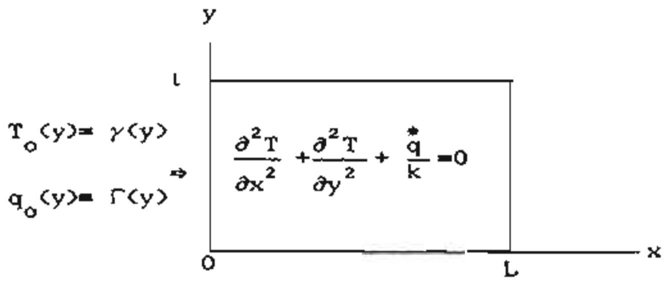

Fig. 1 The stated inverse problem

Assuming constant thermal conductivity and uniform internal heat generation, the mathematical formulation of the problem may be given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\overset{*}{q}}{k} = 0, \qquad (1a)$$

$$T_o(y) = T(0,y) = \gamma(y), \qquad (1b)$$

$$q_o(y) = q_x(0,y) = -k\frac{\partial T}{\partial x}\Big|_{x=0} = \Gamma(y). \qquad (1c)$$

where $\gamma(y)$ and $\Gamma(y)$ are arbitrary functions of y, which should be continuous and differentiable. $\overset{*}{q}$ is the volumetric heat generation rate.

It is known that the superposition principle is frequently used in heat conduction analysis to solve a more complex problem by dividing it into a number of simple problems; all adding up the posed problem. Based on this knowledge, we simplify the problem of Fig. 1 by dividing it into two simpler problems (cf Fig. 2).

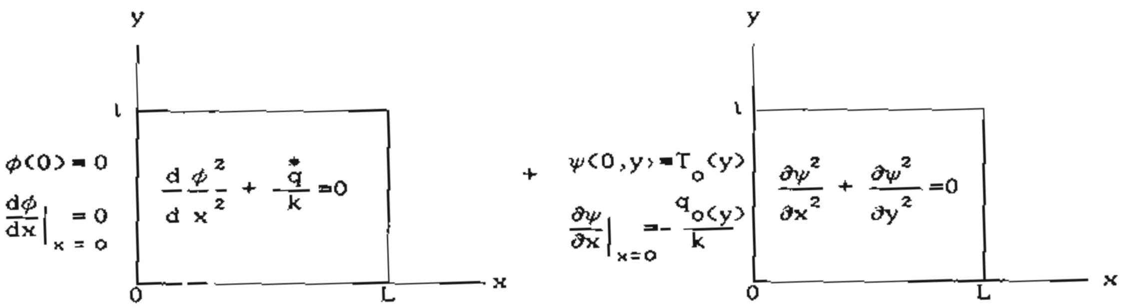

Fig. 2 Dividing problem (1) into 2 simpler problems

By this way, solution of problem (1) is assumed to be

$$T(x,y) = \psi(x,y) + \phi(x) \qquad (2)$$

Here, $\phi(x)$ is assumed to satisfy the one-dimensional problem :

$$\frac{d^2\phi}{dx^2} + \frac{\overset{*}{q}}{k} = 0, \qquad (3a)$$

$$\phi(0) = 0, \qquad (3b)$$

$$\frac{d\phi}{dx}\Big|_{x=o} = 0. \qquad (3c)$$

From combining Eqs. (1a) to (3c), we find that the function $\psi(x,y)$ is satisfied by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \qquad (4a)$$

$$\psi(0,y) = T_o(y), \qquad (4b)$$

$$\frac{\partial\psi}{\partial x}\Big|_{x=o} = -\frac{q_o(y)}{k}, \qquad (4c)$$

Hence, the solution of problem (3) is

$$\phi(x) = -\frac{\overset{*}{q}}{2k}x^2 \tag{5}$$

We start analysis problem (4) by assuming the solution is an infinite series in terms of the exterior x-gradient of temperature at the boundary surface (0,y) [9-11] :

$$\psi(x,y) = \sum_{n=0}^{\infty} C_n(x) \left.\frac{\partial^n \psi}{\partial x^n}\right|_{x=0} \tag{6}$$

where $C_n(x)$ is x-dependent coefficient.

Dividing the above series in Eq. (6) into even and odd terms gives

$$\psi(x,y) = \sum_{n=0}^{\infty} C_{2n}(x) \left.\frac{\partial^{2n}\psi}{\partial x^{2n}}\right|_{x=0} + \sum_{n=0}^{\infty} C_{2n+1}(x) \left.\frac{\partial^{2n+1}\psi}{\partial x^{2n+1}}\right|_{x=0} \tag{7}$$

By some mathematical manipulations, the even terms can be substituted by

$$\left.\frac{\partial^{2n}\psi}{\partial x^{2n}}\right|_{x=0} = (-1)^n \frac{d^{2n}T_o(y)}{dy^{2n}} \tag{8}$$

and the odd terms by

$$\left.\frac{\partial^{2n+1}\psi}{\partial x^{2n+1}}\right|_{x=0} = -\frac{(-1)^n}{k}\frac{d^{2n}q_o(y)}{dy^{2n}} \tag{9}$$

Concerning the process of finding the above two relations the reader can refer to reference [9] for further edification. Substituting Eqs. (8), (9) into Eq. (7) gives

$$\psi(x,y)=\sum_{n=0}^{\infty} a_n(x)\frac{d^{2n}T_o(y)}{dy^{2n}} - \frac{1}{k}\sum_{n=0}^{\infty} b_n(x)\frac{d^{2n}q_o(y)}{dy^{2n}} \tag{10}$$

where we introduced $a_n(x) = (-1)^n C_{2n}(x)$ and $b_n(x) = (-1)^n C_{2n+1}(x)$. Up to this point, the remaining task is to determine the $a_n(x)$ and $b_n(x)$ functions. From the requirement that Eq. (10) exactly satisfies Eqs. (4a)-(4c), these functions are found to be expressed by

$$\nabla^2 a_o(x)= 0, \qquad \nabla_x^2 a_n(x)= - a_{n-1}(x); \quad n = 1,2,3... \tag{11}$$

$$\nabla^2 b_o(x)= 0, \qquad \nabla_x^2 b_n(x)= - b_{n-1}(x); \quad n = 1,2,3.., \tag{12}$$

with the boundary conditions

$$a_o(0)=1, \quad b_o(0)=0, \quad \text{and} \quad a_n(0)=b_n(0)=0, \tag{13}$$

$$\nabla_x b_o(0)=1, \quad \nabla_x a_o(0)=0 \quad \text{and} \quad \nabla_x a_n(0)=\nabla_x b_n(0)=0; \quad n=1,2,3,.... \tag{14}$$

where $\nabla_x^2 = d^2/dx^2$ and $\nabla_x = d/dx$. The solution to Eqs. (11) and (12) subject to boundary conditions (13) and (14) completely determines the $a_n(x)$ and $b_n(x)$ functions which can be expressed by

$$a_n(x) = \frac{(-1)^n x^{2n}}{(2n)!} \quad (15), \qquad b_n(x) = \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{16}$$

From Eqs. (15), (16) into Eq. (10) one obtains the solution of problem (4) as

$$\psi(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n}T_o(y)}{dy^{2n}} - \frac{1}{k}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \frac{d^{2n}q_o(y)}{dy^{2n}} \tag{17}$$

Substituting Eqs. (17) and (5) into Eq. (2) gives the general solution of the main problem (1) :

$$T(x,y)=\left[T_o(y)-\frac{x}{k} \; q_o(y)-\frac{\overset{*}{q}\, x^2}{2k}\right]+\sum_{n=1}^{\infty} a_n(x) \frac{d^{2n}T_o(y)}{dy^{2n}} - \frac{1}{k}\sum_{n=1}^{\infty} b_n(x) \frac{d^{2n}q_o(y)}{dy^{2n}} \tag{18}$$

It is important to note that the terms involved in the brackets represent the one-dimensional solution, while the effect of two-dimensional heat transfer are in the remaining terms. For insulated condition at the boundary surface (0,y), the terms of $q_o(y)$ disappear from the solution, while for an isothermal surface condition, the temperature derivative terms will . . vanish. It is also clear that if there is no heat generation, the solution reduces to that obtained in our previous work [9].

Finally, the heat flux components can be calculated by applying Fourier's law on Eq. (18), This yields :

$$q_x(x,y) = \overset{*}{q}x - k\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \frac{d^{2n}T_o(y)}{dy^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n}q_o(y)}{dy^{2n}} \tag{19}$$

$$q_y(x,y)=-k\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n+1}T_o(y)}{dy^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \frac{d^{2n+1}q_o(y)}{dy^{2n+1}} \tag{20}$$

Different versions of the present solution can be obtained, referred to the same geometry and coordinates system of Fig. 1, if the corresponding two boundary conditions required, are prescribed at another boundary surface rather than the boundary plane (0, y), as it is demonstrated in appendix (I).

3 VALIDITY OF THE METHOD

It is important to consider test problems; all have known exact solution, in order to illustrate the application of the method in more detail as well to prove its validity. Since such inverse test problems are not available to us from literatures, therefore, we use known exact solutions of direct problems to construct test inverse problems. For this purpose, we consider the following exact solution :

$$\frac{T(x,y)}{\overset{*}{q}L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right] - 2\sum_{n=0}^{\infty} C_n \cosh \lambda_n y \cos \lambda_n x \tag{21a}$$

wherein $C_n = \dfrac{(-1)^n}{(\lambda_n L)^3 \cosh \lambda_n l}$ and $\lambda_n = (2n+1)\pi/(2L)$.

The above solution has been derived (in reference [12], page 220-221) for direct problem specified by Eq. (1a) with the 4 boundary conditions:

$$T(L,y) = T(0,y) = 0 \quad \text{and} \quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \tag{21b}$$

Now, by use Eq. (21a) we construct below an inverse problem corresponding to that described in Fig. 1.

Test problem 1: The given boundary conditions are :

$$\frac{T_o(y)}{\overset{*}{q}L^2/k} = \frac{T(0,y)}{\overset{*}{q}L^2/k} = \frac{1}{2} - 2\sum_{n=0}^{\infty} C_n \cosh \lambda_n y \tag{22a}$$

$$q_o(y) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{22b}$$

Eqs. (22a,b) are calculated from Eq. (21a) with x=0. With boundary condition (22b) the general solution, given by Eq. (18), reduces to

$$T(x,y) = T_o(y) - \frac{\overset{*}{q}x^2}{2k} + \sum_{n=1}^{\infty} a_n(x) \frac{d^{2n}T_o(y)}{dy^{2n}} \tag{22c}$$

dividing the two sides of the above equation by $\overset{*}{q}L^2/k$ yields

$$\frac{T(x,y)}{\overset{*}{q}L^2/k} = \frac{T_o(y)}{\overset{*}{q}L^2/k} - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \sum_{n=1}^{\infty} a_n(x) \frac{d^{2n}}{dy^{2n}}\left\{\frac{T_o(y)}{\overset{*}{q}L^2/k}\right\} \tag{22d}$$

By substituting $a_n(x)$ from Eq. (15), and $T_o(y)$ and its derivatives from Eq. (22a) into the above equation, and after collecting the common terms, one obtains

$$\frac{T(x,y)}{\overset{*}{q}L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right] - 2\sum_{n=0}^{\infty} C_n \cosh \lambda_n x \left\{1 - \frac{(\lambda_n x)^2}{2!} + \frac{(\lambda_n x)^4}{4!} - \frac{(\lambda_n x)^6}{6!} + \ldots,\infty\right\} \tag{22e}$$

The expanded infinite series in the end of the right-hand-side of the above equation is just the cosine function expansion. Thus,

Eq. (22e) in its closed form, is just the same exact solution (21a).
Since no another direct problem with known exact solution is available to us from references, to be used for constructing another inverse test problem, therefore, in appendix (II) we have constructed classical direct problem specified with 4 boundary conditions, and found its exact solution :

$$T(x,y) = T_o + x - \frac{\overset{*}{q}x^2}{2k} + \frac{1}{\omega}\sinh \omega x \cos \omega y; \quad \omega=\pi/l. \qquad (23)$$

With help of the above exact solution we construct the following inverse test problem no. 2.

Test problem 2 : The known boundary conditions are :

$$T_o(y)(=T(0,y))=T_o \qquad (24a) \text{ and } \qquad q_o(y)/k=-(1+\cos \omega y) \qquad (24b)$$

Under the boundary conditions (24a)the general solution, given by Eq. (18), reduces to

$$T(x,y) = \left[T_o - \frac{x}{k} q_o(y) - \frac{\overset{*}{q}x^2}{2k}\right] - \frac{1}{k}\sum_{n=1}^{\infty} a_n(x) \frac{d^{2n}q_o(y)}{dy^{2n}} \qquad (25)$$

From substituting $a_n(x)$ and $q_o(y)$ from Eqs. (15) and (24b), respectively, into Eq. (25) and after collecting the common terms, one obtains:

$$T(x,y) = T_o + x - \frac{\overset{*}{q}x^2}{k} + \frac{1}{\omega}\cos\omega y \sum_{n=0}^{\infty} \frac{(\omega x)^{2n+1}}{(2n+1)!} \qquad (26)$$

The infinite series in the right of Eq. (26) is just the expansion of the hyperbolic sine function. Thus, Eq. (26) in closed form is the same exact solution (cf Eq. (23)) derived in appendix (II) by 4 conditions. This proves validity of our method.
Rather two test problems, constructed with use of the above exact solution, are presented in appendix (I), both prove the validity of the method and its independence on the kind of the boundary conditions at the other three boundaries.

4 CONCLUSION

This work is concerned with estimating 2-D steady temperature field within a planar geometry involving internal heat generation, by utilizing the temperature and the exterior heat flux distributions; both specified at the same boundary surface as continuous and differentiable functions of the spatial coordinate along the surface. The resultant general solution is explicit, exact and independent on kind of the boundary conditions at the other three boundary surfaces. The effect of internal heat generation is exactly modeled in the solution. The method may also be considered of a practical interest, however, to some steady heat transfer experiments with a flat plate as the test section, if two boundary conditions at the same boundary surface, gained from measurements, fall in one of the following categories:
a-isothermal surface with known heat flux profile
b-insulated surface with known temperature profile
c-both the heat flux and temperature distributions are known

APPENDIX (I)

This appendix summarizes rather two different versions of the present general solution, which are corresponding to case 1 and case 2 of the posed problem described in Fig. 3. Two test problems, constructed by use the exact solution of appendix (II), are given to prove validity of both versions.

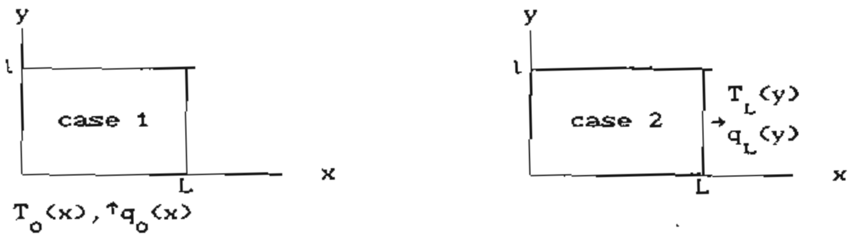

Fig. 3 Other features for the posed problem

Solution of case 1 :

$$T(x,y)=\left[T_o(x)-\frac{y}{k}q_o(x)-\frac{\overset{*}{q}\,y^2}{2k}\right]+\sum_{n=1}^{\infty} f_n(y)\frac{d^{2n}T_o(x)}{dx^{2n}}-\frac{1}{k}\sum_{n=1}^{\infty} g_n(y)\frac{d^{2n}q_o(x)}{dx^{2n}} \qquad (27a)$$

wherein

$$f_n(y)=\frac{(-1)^n\,y^{2n}}{(2n)!} \quad (27b), \qquad g_n(y)=\frac{(-1)^n\,y^{2n+1}}{(2n+1)!} \quad (27c)$$

$T_o(x)=T(x,0)$ and $q_o(x)=q_y(x,0)$, both are continuous and differentiable functions of x.

Test problem : The known 2 boundary conditions are:

$$T_o(x)=T_o+x-\frac{\overset{*}{q}\,x}{2k}+(\sinh\omega x)/\omega \quad (28a), \text{ and } \qquad q_o(x)=0 \quad (28b)$$

The two boundary conditions are calculated by known exact solution, (23), The problem represents case 1 of Fig 3. According to the boundary condition (28b) the general solution, given by Eq. (27a), reduces to

$$T(x,y)=T_o(x)-\frac{\overset{*}{q}\,y^2}{2k}+\sum_{n=1}^{\infty} f_n(y)\frac{d^{2n}T_o(x)}{dx^{2n}} \qquad (29)$$

By substituting $T_o(x)$ and $f_n(y)$ from Eqs. (28a) and (27b), respectively, into Eq. (29), and after collecting the common terms, one obtains :

$$T(x,y)=T_o+x-\frac{\overset{*}{q}\,x^2}{2k}+\frac{1}{\omega}\sinh\omega x\sum_{n=0}^{\infty}\frac{(-1)^n(\omega y)^{2n}}{(2n)!} \qquad (30)$$

The above infinite series is just the cosine function expansion, thus, Eq. (30) in its closed form, is just the same known exact solution (cf Eq.(23)). It is noted that the unused boundary conditions on the surfaces (0,y) and (L,y) are not homogeneous. This proves the validity of the proposed method and its independent on the kind of the other boundary conditions.

Solution of case 2 :

$$T(x,y)=\left[T_L(y)-\frac{(x-L)}{k}q_{x=L}-\frac{\overset{*}{q}}{2k}(x-L)^2\right]+\sum_{n=1}^{\infty}d_n(x)\frac{d^{2n}T_L(y)}{dy^{2n}}-\frac{1}{k}\sum_{n=1}^{\infty}e_n(x)\frac{d^{2n}q_L(y)}{dy^{2n}} \quad (31a)$$

where

$$d_n(x)=\frac{(-1)^n(x-L)^{2n}}{(2n)!} \quad (31b), \qquad e_n(x)=\frac{(-1)^n(x-L)^{2n+1}}{(2n+1)!} \quad (31c)$$

$T_L(y)=T(L,y)$ and $q_L(y)=q_x(L,y)$ are continuous functions of y.

Test problem : The known boundary conditions are:

$$T_L(y)(=T(L,y)) = T_o+L-\frac{\overset{*}{q}L^2}{2k}+\frac{1}{\omega}\sinh\omega L\cos\omega y \quad (32a)$$

$$q_L(y)(=q_x(L,y))= k\left[\frac{\overset{*}{q}L}{2k}-\cos\omega y\cosh\omega L-1\right] \quad (32b)$$

This problem represents case 2 of Fig. 3. The general solution is Eq. (31a). By substituting from Eqs. (32a) and (32b) into Eq. (31a), and after some mathematical abbreviations with collecting the common terms, we obtain

$$T(x,y)=T_o+x-\frac{\overset{*}{q}x^2}{2k}+\frac{1}{\omega}\cos\omega y\left[\sinh\omega L\sum_{n=0}^{\infty}\frac{(\omega(x-L))^{2n}}{(2n)!}+\cosh\omega L\sum_{n=0}^{\infty}\frac{(\omega(x-L))^{2n+1}}{(2n+1)!}\right] \quad (33)$$

From the right of the above equation, it can be noted that the first series is just the expansion of the hyperbolic cosine function, and the second series is the expansion of the hyperbolic sine function. Thus, Eq. (33) can be expressed in the closed form:

$$T(x,y)=T_o+x-\frac{\overset{*}{q}x^2}{2k}+\frac{1}{\omega}\cos\omega y\left[\sinh\omega L\cosh\omega(x-L)+\cosh\omega L\sinh\omega(x-L)\right]$$

$$=T_o+x-\frac{\overset{*}{q}x^2}{2k}+\frac{1}{\omega}\cos\omega y\sinh\omega x \quad (34)$$

Eq. (34) is the same exact solution (cf Eq.(23)).

APPENDIX (II)

Here, we construct 2-dimensional, steady heat conduction problem from the direct type, and then derive its solution. This problem is described by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q^*}{k} = 0, \tag{35a}$$

$T(0,y)=T_o$ (35b) $\quad \left.\frac{\partial T}{\partial x}\right|_{x=o} = 1 + \cos(\pi y/l)$ (35c)

$\left.\frac{\partial T}{\partial y}\right|_{y=o} = 0$ (35d) $\quad \left.\frac{\partial T}{\partial y}\right|_{y=l} = 0$ (35e)

where T_o is constant. Applying the principle of superposition, solution of the above problem may be assumed to be

$$T(x,y) = T_1(x) + T_2(x,y) \tag{36}$$

Assuming $T_1(x)$ is satisfied the one-dimensional problem :

$$\frac{d^2T_1}{dx^2} + \frac{q^*}{k} = 0, \quad \text{with} \quad T_1(0) = T_o, \quad \left.\frac{dT_1}{dx}\right|_{x=o} = 1 \tag{37}$$

we find that $T_2(x,y)$ is satisfied by the problem :

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0, \tag{38a}$$

$\left.\frac{\partial T_2}{\partial y}\right|_{y=o} = 0$ (38b) $\quad \left.\frac{\partial T}{\partial y}\right|_{y=l} = 0$ (38c)

$T_2(0,y)=0$ (38d) $\quad \left.\frac{\partial T_2}{\partial x}\right|_{x=o} = \cos(\pi y/l)$ (38e)

By separation of variables , the general solution of problem (38) may be assumed by

$$T_2(x,y) = X(x)\, Y(y) \tag{39}$$

Substituting Eq. (39) into Eq. (38a) yields

$$\frac{1}{X}\frac{d^2X}{d\,x^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{40}$$

The two sides of the above equation can be equal, only if both are equal a constant value, say ω^2

$\frac{d^2Y}{dy^2} + \omega^2 Y = 0$ (41) $\quad \frac{d^2X}{dx^2} - \omega^2 X = 0$ (42)

The general solution, from those of Eqs.(41) and (42),is assumed

$$T_2(x,y) = (C_1 e^{-\omega x} + C_2 e^{\omega x})(C_3 \cos \omega y + C_4 \sin \omega y) \quad (43)$$

The boundary condition (38b) with Eq. (43) gives $C_4 = 0$, and Eq. (38d) gives $C_1 = -C_2$. Consequently, Eq.(43) becomes

$$T_2(x,y) = C \sinh \omega x \cos \omega y \quad (44)$$

where we introduced $C = 2C_3C_2$. Applying the boundary condition (38c) on Eq. (46) yields : $0 = C \sinh \omega x \sin \omega l$, which holds only if $\sin \omega_n l = 0$ with $\omega_n = n\pi/l$; $n = 1,2,3,\ldots\ldots,\infty$. Thus, Eq. (44) can be expressed by an infinite series:

$$T_2(x,y) = \sum_{n=o}^{\infty} C_n \sinh \omega_n x \cos \omega_n y; \quad \omega_n = n\pi/l \quad (45)$$

The boundary condition (38e) with the above equation gives

$$\cos \pi y/l = \sum_{n=o}^{\infty} C_n \omega_n \cos \omega_n y \quad (46)$$

Eq.(46) holds only, if $C_o = 0$, $C_1 = 1/\omega_1$ and $C_2 = C_3 = \ldots C_n = 0$. Thus, Eq. (46) becomes

$$T_2(x,y) = (\sinh \omega x \cos \omega y)/\omega; \quad \omega = \pi/l \quad (47)$$

The solution of problem (37) is found :

$$T_1(x) = T_o + x - \frac{\overset{*}{q} x^2}{2k} \quad (48)$$

Substituting Eqs. (47) and (48) into Eq. (36) gives

$$T(x,y) = T_o + x - \frac{\overset{*}{q} x^2}{2k} + (\sinh \omega x \cos \omega y)/\omega; \quad \omega = \pi/l. \quad (49)$$

NOMENCLATURES

$a_n(x)$, $d_n(x)$, $f_n(x)$	x-dependent coefficients, m^{2n}
$b_n(x)$, $e_n(x)$, $g_n(x)$	x-dependent coefficients, m^{2n+1}
$c_n(x)$	x-dependent coefficients, m^n
l	plate height, m
L	plate width, m
k	thermal conductivity, $kW/(m\,^oC)$
q	heat flux, kW/m^2
$q_o(y)$	$= q_x(0,y)$, x-direction heat flux outside the boundary surface (0,y), kW/m^2
$q_L(y)$	$= q_x(L,y)$, x-direction heat flux outside the boundary surface (L,y), kW/m^2

$q_o(x)$	$= q_y(x,0)$, y-direction heat flux outside the boundary surface $(x,0)$, kW/m^2
T	temperature, ^oC
T_o	constant, ^oC
$T_o(y)$	$= T(0,y)$, temperature of the boundary surface $(0,y)$, C^o
$T_L(y)$	$= T(L,y)$, temperature of the boundary surface (L,y), ^oC
$T_o(x)$	$= T(x,0)$, temperature of the boundary surface $(x,0)$, ^oC
x, y	cartesian coordinates, m
$\dot{q}$	volumetric heat generation rate, kW/m^3
α	thermal diffusivity, m^2/s
∇^2_x	$= d^2/dx^2$, Laplacian operator, $1/m^2$
$\phi(x)$	x-dependent function, ^oC
$\psi(x,y)$	(x,y)-dependent function, ^oC

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