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CONTINUED FRACTION EVALUATION OF THE NORMAL DISTRIBUTION FUNCTION

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Abstract

In this paper, continued fraction expansion of the normal distribution function is developed. An efficient and simple computational algorithm based on this expansion is also developed using top-down evaluation procedure. Numerical results of the algorithm are in full greement at least to fifteen digits accuracy with that of standard tables.

1. Introduction

Probability theory plays very serious role in many problems of space dynamics. Of these problems are for examples, orbit determination of space objects [e.g. Vallado, 1997] and space navigation problems [Battin, 1999].

One important integral of common appearance in the above and other applications is the normal distribution function $P(x)$ defined as

$$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du,\ x\geq 0. \qquad (1.1)$$

There are several methods available for the evaluation of this integral, all depending on polynomial evaluations with different degrees of accuracy [e.g. Hastings, 1955 and Abramowitz, 1970].

In fact, continued fraction expansions are, generally, far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series.

Due to the above importance of $P(x)$, and on the other hand, due to the efficiency of continued fraction for evaluating functions are what motivated our work : to establish computational algorithm for the function $P(x)$ based on its continued fraction expansion.

2. Basic Formulations

2.1. $P(x)$ IN TERMS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

Recalling Equation (1.1) we have

$$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}u^2}du, \quad x\geq 0,$$

since $x\geq 0$, then

$$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}u^2}du-\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-\frac{1}{2}u^2}du+\frac{1}{\sqrt{2\pi}}\int_{0}^{x}e^{-\frac{1}{2}u^2}du,$$

or

$$P(x)=1-\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{0}^{x}e^{-\frac{1}{2}u^2}du=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{0}^{x}e^{-\frac{1}{2}u^2}du. \tag{2.1}$$

Since

$$e^{-\frac{1}{2}u^2}=\sum_{j=0}^{\infty}\frac{(-1)^j u^{2j}}{2^j j!},$$

we obtain from Equation (2.1) that

$$P(x)=\frac{1}{2}+\frac{x}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{1}{j!(2j+1)}\left(-\frac{x^2}{2}\right)^j.$$

This equation could be written as

$$P(x)=\frac{1}{2}+\frac{x}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{(1/2)\left\{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2j-1}{2}\right)\right\}}{j!\left\{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2j-1}{2}\right)\right\}\left(\frac{2j+1}{2}\right)}\left(-\frac{x^2}{2}\right)^j.$$

or

$$P(x)=\frac{1}{2}+\frac{x}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{(1/2)_j}{j!\left(\frac{3}{2}\right)_j}\left(-\frac{x^2}{2}\right)^j, \tag{2.2}$$

where

$$(\eta)_j=\eta(\eta+1)(\eta+2)\ldots(\eta+j-1) \quad ;(\eta)_0=1.$$

From Equation (2.2) it follows that

$$P(x)=\frac{1}{2}+\frac{x}{\sqrt{2\pi}}M\left(\frac{1}{2},\frac{3}{2};\frac{1}{2}x^2\right), \tag{2.3}$$

where $M(\beta,\gamma;z)$ is called a *confluent hypergeometric function* defined in terms of the *hypergeometric function* $F(\alpha,\beta,\gamma;x)$ as

$$M(\beta,\gamma;z)=\lim_{\alpha\to\infty}F(\alpha,\beta,\gamma;z/\alpha), \tag{2.4}$$

that is

$$M(\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}. \tag{2.5}$$

According to Kummer transformation [e.g. Abramowitz, 1970], which is

$$M(\beta,\gamma;z)=e^{z}M(\gamma-\beta,\gamma,-z),$$

then Equation (2.3) could be written as

$$P(x)=\frac{1}{2}+\frac{x}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}M\left(1,\frac{3}{2};\frac{1}{2}x^2\right). \tag{2.6}$$

2.2. CONTINUED FRACTION EXPANSION OF $P(x)$

The confluent hypergeometric functions satisfy the identities [Erdelyi, 1953];

$$M(\beta+1,\gamma+1;x)-M(\beta,\gamma;x)=\frac{\gamma-\beta}{\gamma(\gamma+1)}xM(\beta+1,\gamma+2;x), \tag{2.7}$$

$$M(\beta+1,\gamma+2;x)-M(\beta+1,\gamma+1;x)=-\frac{(\beta+1)}{(\gamma+1)(\gamma+2)}xM(\beta+2,\gamma+3;x). \tag{2.8}$$

Consider the following sequence of confluent hypergeometric functions defined for n=0,1,2,...

$$M_{2n}=M(\beta+n,\gamma+2n;x), \tag{2.9}$$

$$M_{2n+1}=M(\beta+n+1,\gamma+2n+1;x)\ . \tag{2.10}$$

From identities (2.7) and (2.8) we have

$$M_{2n+1}-M_{2n}=\delta_{2n+1}xM_{2n+2}, \tag{2.11}$$

$$M_{2n}-M_{2n-1}=\delta_{2n}xM_{2n+1}, \tag{2.12}$$

where the odd-and even-labelled δ's are determined from

$$\delta_{2n+1}=\frac{\gamma-\beta+n}{(\gamma+2n)(\gamma+2n+1)}\ , \tag{2.13}$$

$$\delta_{2n}=-\frac{\beta+n}{(\gamma+2n)(\gamma+2n+1)}\ . \tag{2.14}$$

Divide Equation (2.11) by M_{2n} and divide Equation (2.12) by M_{2n-1} and define

$$G_{2n}=\frac{M_{2n+1}}{M_{2n}}\ , \tag{2.15}$$

$$G_{2n-1}=\frac{M_{2n}}{M_{2n-1}}\ , \tag{2.16}$$

then we get

$$G_{2n}-1=\delta_{2n+1}xG_{2n+1}G_{2n} \quad ,$$

$$G_{2n-1}-1=\delta_{2n}xG_{2n}G_{2n-1}$$

or

$$G_{2n}=\frac{1}{1-\delta_{2n+1}xG_{2n+1}} \quad ,$$

$$G_{2n-1}=\frac{1}{1-\delta_{2n}xG_{2n}} .$$

If we put successively n=0, n=1, etc., we derive a continued fraction expansion for $G_0=M_1/M_0$. Thus

$$\frac{M(\beta+1,\gamma+1;x)}{M(\beta,\gamma;x)}=\cfrac{1}{1-\cfrac{\delta_1 x}{1-\cfrac{\delta_2 x}{1-\cfrac{\delta_3 x}{\ddots \atop 1-\delta_{2n}xG_{2n}}}}} \tag{2.17}$$

and letting n become infinite results in an infinite continued fraction.

Now, since $M(0,\gamma;x)=1$, then the continued fraction of Equation (2.17) represents the function $M(1,\gamma+1;x)$. Therefore, if we replace γ by $\gamma-1$, we get

$$M(1,\gamma;x)=\cfrac{1}{1-\cfrac{\beta_1 x}{1-\cfrac{\beta_2 x}{1-\cfrac{\beta_3 x}{1-\ddots}}}} \tag{2.18}$$

where

$$\beta_{2n+1}=\frac{\gamma+n-2}{(\gamma+2n-1)(\gamma+2n)},\beta_{2n}=\frac{-n}{(\gamma+2n-1)(\gamma+2n-2)} . \tag{2.19}$$

Finally, from Equations (2.18), (2.19) and (2.6), we get for P (x) the required continued fraction expansion as

$$P(x)=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.\cfrac{x}{1-\cfrac{x^2}{3+\cfrac{2_2x^2}{5-\cfrac{3x^2}{7+\cfrac{4x^2}{9-\ddots}}}}} \tag{2.20}$$

3.Computational Developments

3.1. TOP-DOWN CONTINUED FRACTION EVALUATION

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n'th convergent were accumulated separately with three-term recurrence formulae. The drawback to the first method, obviously, has to decide far down the fraction to begin in order to ensure convergence. The drawback to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm, which works from top down while avoiding numerical difficulties, would be ideal from a programming standpoint.

Gautschi [1967] proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is given as

$$c = \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \cfrac{n_4}{\ddots}}}} \equiv \frac{n_1}{d_1 +}\frac{n_2}{d_2 +}\frac{n_3}{d_3 +}\cdots$$

then initialize the following parameters

$$a_1 = 1,$$

$$b_1 = n_1/d_1,$$

$$c_1 = n_1/d_1,$$

and iterate (k=1,2,...) according to

$$a_{k+1} = \frac{1}{1+\left[\frac{n_{k+1}}{d_k d_{k+1}}\right] a_k},$$

$$b_{k+1} = [a_{k+1} - 1] b_k,$$

$$c_{k+1} = c_k + b_{k+1} .$$

In the limit, the c sequence converges to the value of the continued fraction.

3.2. COMPUTATIONAL ALGORITHM

- **Purpose :** To compute the value of the normal distribution function $P(x)$ for a given value of $x \geq 0$, using continued fraction expansion of Equation (2.20).

- **Input :** x, ε: Tolerance specifies the upper bound of the absolute value of b [in Gautschi's algorithm of Subsection 3.1] and below which the calculations are terminated, N: The maximum number of the recurrent calculations.

- **Output** : $P(x)$, IER: Resultant error parameter coded as follows

 * IER=0,means that, the convergence in computing c is achieved in a number of cycles $\leq N$ within accuracy specified by ε

 * IER=1 means that, no convergence after N cycles is achieved within the accuracy specified by ε.

- **Computational sequence**

1-Set IER = 0

2-Set $n = x, a = 1, b = n, d_1 = 1, c = n$

3-If $|b| \leq \varepsilon$ go to step **6**

4-For i=2,3,...,N, do:

- $n = (-1)^{i-1} * (i-1) * x^2$
- $d_2 = 2 * i - 1$
- $a = \dfrac{1}{1 + \left(\dfrac{n}{d_1 * d_2}\right) a}$
- $b = (a-1) * b$
- $c = c + b$
- If $|b| \le \varepsilon$ go to step 6
- $d_1 = d_2$

1- IER = 1

2- $P(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} * c$

3- End

3.3. NUMERICAL RESULTS

The above computational algorithm was applied to construct Table I for $P(x), x = 0.02(0.02)5$ up to fifteen digits accuracy. Within this accuracy, the present results agree completely with those given in [Abramowitz, 1970].

In concluding the present paper, an efficient and simple computational algorithm for the normal distribution function $P(x), x \ge 0$ was established using continued fraction expansion. Comparisons with the standard tables show that, the results of the algorithm are in full agreement at least to fifteen digits accuracy.

TABLE I

Values Of The Normal Distribuion Function $P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}} dx$ **For** $x = .02\ (.02)\ 5$

Using Continued Fraction Evaluation

x	P (x)	x	P (x)	x	P (x)
0.02	0.50797 83127 16902	1.02	0.84613 57696 27265	2.02	0.97830 83062 32353
0.04	0.51595 34368 52831	1.04	0.85083 00496 69019	2.04	0.9[illegible]32 [illegible]8371 3393
0.06	0.52392 21826 54107	1.06	0.85542 77003 36090	2.06	0.98030 07295 90623
0.08	0.53188 13720 13987	1.08	0.85992 89099 11231	2.08	0.98123 72335 65062
0.10	0.53982 78372 77029	1.10	0.86433 39390 53617	2.10	0.98213 55794 37183
0.12	0.54775 84260 20584	1.12	0.86864 31189 57269	2.12	0.98299 69773 52368
0.14	0.55567 00048 05906	1.14	0.87285 68494 37202	2.14	0.98382 26166 27834
0.16	0.56355 94628 91433	1.16	0.87697 55969 48657	2.16	0.98461 36652 16074
0.18	0.57142 37159 00901	1.18	0.88099 98925 44799	2.18	0.98537 12692 24011
0.20	0.57925 97094 39103	1.20	0.88493 03297 78292	2.20	0.98609 65524 86502
0.22	0.58706 44226 48215	1.22	0.88876 75625 52166	2.22	0.98679 06161 92743
0.24	0.59483 48716 97796	1.24	0.89251 23029 25413	2.24	0.98745 45385 64055
0.26	0.60256 81132 01761	1.26	0.89616 53188 787	2.26	0.98808 93745 81453
0.28	0.61026 12475 55797	1.28	0.89972 74320 45558	2.28	0.98869 61557 61447
0.30	0.61791 14221 88953	1.30	0.90319 95154 1439	2.30	0.98927 58899 78324
0.32	0.62551 58347 2332	1.32	0.90658 24910 06528	2.32	0.98982 95613 31281
0.34	0.63307 17360 36029	1.34	0.90987 73275 35547	2.34	0.99035 81300 54642
0.36	0.64057 64332 17991	1.36	0.91308 50380 52915	2.36	0.99086 25324 69428
0.38	0.64802 72924 24163	1.38	0.91620 66775 84986	2.38	0.99134 36809 74483
0.40	0.65542 17416 10324	1.40	0.91924 33407 66229	2.40	0.99180 24640 75404
0.42	0.66275 72731 51751	1.42	0.92219 61594 73454	2.42	0.99223 97464 49447
0.44	0.67003 14463 39406	1.44	0.92506 63004 65673	2.44	0.99265 63690 44652
0.46	0.67724 18897 49652	1.46	0.92785 49630 34106	2.46	0.99305 31492 11376
0.48	0.68438 63034 83777	1.48	0.93056 33766 66668	2.48	0.99343 08808 64453
0.50	0.69146 24612 74013	1.50	0.93319 27987 31142	2.50	0.99379 03346 74224
0.52	0.69846 82124 53034	1.52	0.93574 45121 81064	2.52	0.99413 22582 84667
0.54	0.70540 14837 84302	1.54	0.93821 98232 88188	2.54	0.99445 73765 56917

TABLE I (Continued)

0.56	0.71226 02811 50973	1.56	0.94062 00594 05207	2.56	0.99476 63918 36445
0.58	0.71904 26911 01436	1.58	0.94294 65667 62246	2.58	0.99505 99842 4223
0.60	0.72574 68822 49926	1.60	0.94520 07083 00442	2.60	0.99533 88119 76281
0.62	0.73237 11065 31017	1.62	0.94738 38615 45748	2.62	0.99560 35116 51879
0.64	0.73891 37003 07139	1.64	0.94949 74165 25897	2.64	0.99585 46986 38965
0.66	0.74537 30853 28664	1.66	0.95154 27737 33277	2.66	0.99609 29674 25147
0.68	0.75174 77695 4643	1.68	0.95352 13421 3628	2.68	0.99631 88919 90826
0.70	0.75803 63477 76927	1.70	0.95543 45372 41457	2.70	0.99653 30261 96959
0.72	0.76423 75022 20749	1.72	0.95728 37792 08672	2.72	0.99673 59041 8411
0.74	0.77035 00028 35209	1.74	0.95907 04910 21194	2.74	0.99692 80407 81348
0.76	0.77637 27075 62401	1.76	0.96079 60967 12517	2.76	0.99710 99319 23774
0.78	0.78230 45624 14267	1.78	0.96246 20196 51483	2.78	0.99728 20550 77299
0.80	0.78814 46014 16603	1.80	0.96406 96808 87074	2.80	0.99744 48696 69572
0.82	0.79389 19464 14187	1.82	0.96562 04975 5411	2.82	0.99759 88175 25813
0.84	0.79954 58067 3955	1.84	0.96711 58813 40836	2.84	0.99774 43233 08458
0.86	0.80510 54787 48192	1.86	0.96855 72370 19247	2.86	0.99788 17949 59594
0.88	0.81057 03452 23288	1.88	0.96994 59610 388	2.88	0.99801 16241 45103
0.90	0.81593 98746 53241	1.90	0.97128 34401 83998	2.90	0.99813 41866 99616
0.92	0.82121 36203 85628	1.92	0.97257 10502 96163	2.92	0.99824 98430 71323
0.94	0.82639 12196 61375	1.94	0.97381 01550 59547	2.94	0.99835 89387 65842
0.96	0.83147 23925 33162	1.96	0.97500 21048 5178	2.96	0.99846 18047 88264
0.98	0.83645 69406 72308	1.98	0.97614 82356 58491	2.98	0.99855 87580 82659
1.00	0.84134 47460 68543	2.00	0.97724 98680 51821	3.00	0.99865 01019 6837

TABLE I (Continued)

3.02	0.99873 61265 72327	4.02	0.99997 09009 29242
3.04	0.99881 71092 56893	4.04	0.99997 32743 99269
3.06	0.99889 33150 42589	4.06	0.99997 54636 42025
3.08	0.99896 49970 25198	4.08	0.99997 74821 49654
3.10	0.99903 23967 8678	4.10	0.99997 93424 9328
3.12	0.99909 57448 00174	4.12	0.99998 10563 80107
3.14	0.99915 52608 2654	4.14	0.99998 26347 08961
3.16	0.99921 11543 05622	4.16	0.99998 40876 20568
3.18	0.99926 36247 38449	4.18	0.99998 54245 45275
3.20	0.99931 28620 62084	4.20	0.99998 66542 51005
3.22	0.99935 90470 1634	4.22	0.99998 77846 84052
3.24	0.99940 23515 0206	4.24	0.99998 88240 10671
3.26	0.99944 29389 30975	4.26	0.99998 97786 54652
3.28	0.99948 09645 66797	4.28	0.99999 06553 34704
3.30	0.99951 65758 57621	4.30	0.99999 14600 94947
3.32	0.99954 99127 59408	4.32	0.99999 21985 39853
3.34	0.99958 11080 50556	4.34	0.99999 28758 65009
3.36	0.99961 02876 37421	4.36	0.99999 34968 78136
3.38	0.99963 75708 50969	4.38	0.99999 40660 35217
3.40	0.99966 30707 34332	4.40	0.99999 45874 57555
3.42	0.99968 68943 21416	4.42	0.99999 50649 56734
3.44	0.99970 91429 06711	4.44	0.99999 55020 58599
3.46	0.99972 99123 06029	4.46	0.99999 59020 19636
3.48	0.99974 92931 08716	4.48	0.99999 62678 50909
3.50	0.99976 73709 20964	4.50	0.99999 66023 31649
3.52	0.99978 42266 00715	4.52	0.99999 69080 24457
3.54	0.99979 99364 83976	4.54	0.99999 71872 96338

TABLE I (Continued)

3.56	0.99981 45726 03083	4.56	0.99999 74423 30232
3.58	0.99982 82028 96257	4.58	0.99999 76751 33727
3.60	0.99984 08914 09856	4.60	0.99999 78875 64871
3.62	0.99985 26984 92109	4.62	0.99999 80813 26369
3.64	0.99986 36809 79559	4.64	0.99999 82579 89269
3.66	0.99987 38923 75844	4.66	0.99999 84189 97936
3.68	0.99988 33830 23181	4.68	0.99999 85656 84509
3.70	0.99989 22002 66519	4.70	0.99999 86992 71044
3.72	0.99990 03886 11031	4.72	0.99999 88208 85363
3.74	0.99990 79898 72517	4.74	0.99999 89315 45835
3.76	0.99991 50433 2149	4.76	0.99999 90322 18726
3.78	0.99992 15855 20649	4.78	0.99999 91237 68579
3.80	0.99992 76519 56056	4.80	0.99999 92069 89673
3.82	0.99993 32741 62968	4.82	0.99999 92826 3358
3.84	0.99993 84825 44747	4.84	0.99999 93513 6305
3.86	0.99994 33064 8745	4.86	0.99999 94138 07186
3.88	0.99994 77717 67674	4.88	0.99999 94705 44341
3.90	0.99995 19036 55976	4.90	0.99999 95220 88992
3.92	0.99995 57255 15667	4.92	0.99999 95689 62745
3.94	0.99995 92591 95451	4.94	0.99999 96115 87445
3.96	0.99996 25251 18416	4.96	0.99999 96504 18084
3.98	0.99996 55423 65856	4.98	0.99999 96858 66052
4.00	0.99996 83287 58111	5.00	0.99999 97183 08249

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تقييم دالة التوزيع المعتدل بطريقة الكسر المستمر

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الملخص العربى

تم في هذا البحث تمثيل دالة التوزيع المعتدل بطريقة الكسر المستمر وتم أيضـــاً تشـــييد طريقـــة حسابية فعالة وبسيطة لهذا التمثيل باستخدام طريقة القمة والهبوط للتقييم .كانت النتـــائج العدديـــة للطريقة الحسابية متفقة إلى خمسة عشر رقم على الأقل مع القيم المعطاة في الجداول القياسية.