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On the classification of periodic points of cubic polynomials over Q

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Abstract: In this paper, we introduce a complete parametrisation of cubic polynomials over Q that have a rational periodic point of period 1 and rational periodic points of period 2. Moreover, Some parametrisation of preperiodic points.

keywords: discrete dynamical systems; fixed point; projective space; height functions

1.Introduction

A *(discrete) dynamical system* consists of a set A and a function $g: A \to A$ which maps the set A to itself. This self-mapping authorizes iteration

$$g^m = \underbrace{g \circ g \circ \ldots \circ g}_{m times} = m^{th} iterate of g.$$

(By convention, g^0 indicates the identity map on A). For a given point $p \in A$, the orbit of p is the set

$$\mathcal{O}_g(p) = \mathcal{O}(p) = \{g^m(p): m \geq 0\}.$$

The point p is said to be *periodic point* of g if $g^m(p) = p$ for some $m \geq 1$. The smallest such m is called the *exact period of* p. And it is called *preperiodic point* if some iterate $g^n(p)$ is periodic. The sets of periodic and preperiodic points of g in A are respectively denoted by

$$Per(g, A) = \{p \in A: g^m(p) = p, m \geq 1\}$$
$$PrePer(g, A) = \{p \in A: g^{n+m}(p) = g^n(p)$$
$$, n \geq 1, m \geq 1\}$$
$$= \{p \in A: \mathcal{O}_g(p) is finite\}$$

When the set A is fixed, we write Per(g) and PrePer(g) instead of $Per(g, A)$ and $PrePer(g, A)$ respectively.

Let a morphism $g: \mathbb{P}^n \to \mathbb{P}^n$ of degree at least two defined over a number field F. For $P \in \mathbb{P}^n(\mathrm{F})$, Northcott used height functions to prove that PrePer(g, F) is always finite. Moreover, the latter set can be computed effectively for a given g. These facts have been rediscovered (in varying degrees of generality) by many authors [1], [2], [3].

The following conjecture has been proposed by Morton and Silverman [4].

Conjecture 1 There exists a bound $B = B(D, n, d)$ such that if $\mathrm{F}/\mathbb{Q}$ is a number field of degree D, and a morphism $g: \mathbb{P}^n \to \mathbb{P}^n$ of degree $d \geq 2$ defined over F, then $O(PrePer(g, \mathrm{F})) \leq B$.

The special case $D = 1, n = 1, d = 4$ of the latter conjecture implies the "strong uniform boundedness conjecture" on elliptic curves, [5]. This holds since torsion points of elliptic curves are exactly the preperiodic points of the multiplication-by-2 map, and their x-coordinates are preperiodic points for the degree 4 rational map that gives $x(2P)$ in terms of $x(P)$. For the case of quadratic polynomials over the rational field $\mathbb{Q}$, the following conjecture has been made [6]:

Conjecture 2. If $N \geq 4$, then there is no quadratic polynomial $g(y) \in \mathbb{Q}[y]$ that has a rational point of exact period N.

Conjecture 2 has been verified for $N = 4$ and $N = 5$ (see [10] and [6], respectively). In addition, when $N = 6$, the conjecture holds true under the condition that Birch-Swinnerton-Dyer holds, [14].

2 Rational Periodic Points

Two polynomials $g(y), f(y) \in \mathbb{Q}[y]$, they are said to be *linearly conjugate* over the rational field $\mathbb{Q}$ if there is a linear polynomial $\ell(y)$ such that $f(y) = \ell(g(\ell^{-1}(y)))$. This maps the rational preperiodic points of $g(y)$ bijectively to the rational preperiodic points of $f(y)$. Given a polynomial $g(y) = a_3y^3 + a_2y^2 + a_1y + a_0 \in \mathbb{Q}[y]$ with $a_3 \neq 0$, $g(y)$ is linearly conjugate to a polynomial of the form $g(y) = ay^3 + by + d, \quad a, b, andd \in \mathbb{Q}$. We begin by classifying the polynomials $g(y) = ay^3 + by + d$ with periodic points of period 1, $i.e.$, fixed points. If $\lambda \in \mathbb{Q}$ is such that $g(\lambda) = \lambda$, then one can write

$$g(y) - y = (y - \lambda)(ay^2 + uy + v).$$

Thus,

$$-\lambda a + u = 0, -\lambda u + v = b - 1, -\lambda v = d.$$

Setting $U = -\lambda u, V = -\lambda v$, one then obtains the following result.

Theorem 2.1 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has a rational periodic point of period* 1 *($i.e.$, a rational fixed point) if and only if $a = -U/\lambda^2$, $b = U - (V/\lambda) + 1$, and $d = V$ for some λ, U and $V \in \mathbb{Q}$. In this case, λ is a rational fixed point of $g(y)$.*

Moreover, If V is given by $V = (\lambda^2U^2 - W^2)/(4\lambda U)$, then we have another two points y_1 and y_2 with period 1 which are given by $y_1 = (-\lambda U + W)/(2U), andy_2 = (-\lambda U + W)/(2U)$ for some $\lambda, U, W \in \mathbb{Q}$. In this case the three point will be distinct if and only if $W \neq \pm 3\lambda U$ or $W \neq 0$.

Theorem 2.2 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has a rational periodic point of exact period two if and only if*

$$a = -\frac{u_3^2}{u_1^2 + u_1u_2 + u_2^2 + u_3u_4}$$

$$b = -\frac{u_3u_4}{u_1^2 + u_1u_2 + u_2^2 + u_3u_4}$$

$$d = \frac{(u_1 + u_2)(u_1^2 + u_2^2 + u_3u_4)}{u_3(u_1^2 + u_1u_2 + u_2^2 + u_3u_4)}$$

for some distinct u_1, u_2, u_3, and $u_4 \in \mathbb{Q}$, $u_3 \neq 0, u_1^2 + u_1u_2 + u_2^2 + u_3u_4 \neq 0$. In this case, there are two rational points, $p_1 = u_1/u_3$ and $p_2 = u_2/u_3$ and these two are cyclically permuted by the function $g(y)$.

Proof. To classify cubic polynomials with periodic points of period two. If p_1 and p_2 are two distinct rational numbers such that $g(p_1) = p_2$ and $g(p_2) = p_1$, then $p_2 = g(p_1) = ap_1^3 + bp_1 + d$ implies that $c = p_2 - ap_1^3 - bp_1$. Now that $p_1 = g^2(p_1) = g(p_2) = ap_2^3 + bp_2 + p_2 - ap_1^3 - bp_1$, one has

$$(p_1 - p_2)(1 + b + a(p_1^2 + p_1p_2 + p_2^2)) = 0.$$

One sets $P_1 = ap_1$, $P_2 = ap_2$. Since $p_1 \neq p_2$, it follows that we need to find a rational point on the following conic C in $\mathbb{Q}[P_1, P_2, a, b, Z]$

$$aZ + ab + P_1^2 + P_1P_2 + P_2^2 = 0.$$

The point $P = (P_1: P_2: a: b: Z) = (0: 0: 0: 0: 1)$ is a rational point that lies on the latter projective conic. In what follows we find a parametrization for the solutions on this conic. One has the rational map $\phi: \mathbb{P}_{\mathbb{Q}}^3 \to C$ such that $\phi(Q)$, where $Q = (u_1: u_2: u_3: u_4: 0)$, is the intersection of the line that joins the points Q and P with C. We assume that the line L spanned by P and Q is given by $\mu Q + \lambda P = (\mu u_1: \mu u_2: \mu u_3: \mu u_4: \lambda)$. Then the intersection with C is given by

$$(P_1, P_2, a, b, Z) = (u_3u_1: u_3u_2: u_3^2: u_3u_4 : -u_1^2 - u_1u_2 - u_2^2 - u_3u_4).$$

So $p_1 = P_1/a = u_1/u_3, p_2 = P_2/a = u_2/u_3$, and $d = p_2 - ap_1^3 - bp_1$.

Theorem 2.3 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(z)$ has rational periodic points of period one and rational periodic points of period two if and only if*

$$a = \frac{u_3^2(u_1 + u_2 - 2qu_3)}{(u_1 - qu_3)(-u_2 + qu_3)(u_1 + u_2 + qu_3)}$$

$$b = [-u_1^3 - u_2^3 + qu_2^2u_3 + q^3u_3^3 + u_1^2(-u_2 + qu_3) + u_1u_2(-u_2 + qu_3)][(u_2 - qu_3) \times (-u_1 + qu_3)(u_1 + u_2 + qu_3)]^{-1}$$

$$d = [q(u_1 + u_2)(u_1^2 - u_1u_2 + u_2^2 - q^2u_3^2)] \times [(u_1 - qu_3)(-u_2 + qu_3)(u_1 + u_2 + qu_3)]^{-1}$$

for some distinct u_1, u_2, and $u_3 \in \mathbb{Q}$, $q \in \mathbb{Q}$, $u_3 \neq 0$, $q \neq u_1/u_3, u_2/u_3$ and $-(u_1 + u_2)/u_3$. In this case, q is the rational periodic point of period 1 also p_1 and p_2 are the rational periodic points of period 2, and where $p_1 = u_1/u_3$ and $p_2 = u_2/u_3$.

Proof. To classify if there exist cubic polynomials with periodic points of period one and period two points. If $g(y) = ay^3 + by + d$ has a rational periodic points of period two, then by Theorem 2.2

$$a = -\frac{u_3^2}{u_1^2 + u_1u_2 + u_2^2 + u_3u_4}$$
$$b = -\frac{u3u4}{u_1^2 + u_1u_2 + u_2^2 + u_3u_4}$$
$$d = \frac{(u_1 + u_2)(u_1^2 + u_2^2 + u_3u_4)}{u_3(u_1^2 + u_1u_2 + u_2^2 + u_3u_4)}$$

Without loss of generality, we may assume that $g(q) = q$, hence

$$q = g(q) = aq^3 + bq + c$$
$$0 = [-u_1^3 - u_1^2u_2 - u_1u_2^2 - u_2^3 + qu_1^2u_3 + qu_1u_2u_3 + qu_2^2u_3 + q^3u_3^3 + u_4u_3(-u_1 - u_2 + 2qu_3)][u_3(u_1^2 + u_1u_2 + u_2^2 + u_3u_4)]^{-1}$$

or

$$u_4 = [u_1^3 + u_1^2u_2 + u_1u_2^2 + u_2^3 - qu_1^2u_3 - qu_1u_2u_3 - qu_2^2u_3 - q^3u_3^3] \times [u_3(-u_1 - u_2 + 2qu_3)]^{-1}$$

3 Preperiodic points

For any two positive integers m and n a rational point p is said to be rational preperiodic point of type m_n for $g(z) \in \mathbb{Q}[z]$ if it gives an m −cycle after n −iterations, we can see the following example, the point $-1/3$ is of type 2_3 for $g(z) = (3/4)z^3 - (19/12)z - 1/6$, since its orbits are

$$-1/3, 1/3, -2/3, 2/3, -1, 2/3, -1, 2/3, \ldots\ldots$$

Theorem 3.1 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has rational preperiodic points of type 1_1 if and only if*

$$a = -\frac{U}{\lambda^2}$$
$$b = \frac{U(\mu^2 + \mu\lambda + \lambda^2)}{\lambda^2}$$
$$d = \frac{\lambda^2 - U\lambda\mu - U\mu^2}{\lambda}$$

for some distinct μ, λ, and $U \in \mathbb{Q}$, $\lambda \neq 0$. In this case, μ is the rational periodic point of period 1_1.

Proof. To classify cubic polynomials with preperiodic points of type 1_1. If $g(y) = ay^3 + by + d$ has a rational preperiodic point μ of type 1_1, then by Theorem 2.1 $a = -U/\lambda^2$, $b = U - (V/\lambda) + 1$,and $d = V$. Without loss of generality, we may assume that $f(\mu) = \lambda$, Hence $\lambda = V + \mu(1 + U - V/\lambda) - (\mu^3U)/\lambda^2$ or $V = -\mu U - (\mu^2U)/\lambda + \lambda$.

Theorem 3.2 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has rational preperiodic points of type 1_2 if and only if*

$$a = \frac{\lambda(-\mu^3 + \lambda^2\nu + \mu^2\nu - \nu^3 + \lambda\mu(-\mu + \nu))}{(\lambda - \nu)(-\mu + \nu)(\lambda + \mu + \nu)}$$
$$b = \frac{\lambda - \mu}{((\lambda - \nu)(-\mu + \nu)(\lambda + \mu + \nu)}$$
$$d = \frac{-\lambda^3 + \mu^3}{(\lambda - \nu)(-\mu + \nu)(\lambda + \mu + \nu)}$$

In this case, ν is the rational periodic point of period 1_2.

Proof. To classify cubic polynomials with preperiodic points of type 1_2. If $g(y) = ay^3 + by + d$ has a rational preperiodic point μ of type 1_2, then by Theorem 3.1 $a = -U/\lambda^2$,$b = (U(\mu^2 + \mu\lambda + \lambda^2))/\lambda^2$, and $d = -\mu U - (\mu^2U)/\lambda + \lambda$. Without loss of generality, we may assume that $f(\nu) = \mu$, Hence $\mu = (-\nu^3U + \mu^2U(\nu - \lambda) + \mu U(\nu - \lambda)\lambda + \nu U\lambda^2 + \lambda^3)/\lambda^2$ or

$$U = -\frac{\lambda^2(\lambda - \mu)}{(\lambda - \nu)(-\mu + \nu)(\lambda + \mu + \nu)}.$$

Theorem 3.3 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has rational preperiodic*

points of type 1_3 if and only if

$$a = [-v^3 - u^2w + u(v^2 + w^2)][(u - v)v(u - w)(v - w)w]^{-1}$$

$$b = [(-u^2v^2 + v^4 + u^3w - uw^3)(2u^6w^2 - u^5(4v^2w + 9w^3) + v^5(2v^3 - 9vw^2 + 9w^3) + u^4(2v^4 + 9v^3w + 9vw^3 + 14w^4) + uv^2(-9v^5 + 9v^4w + 9v^3w^2 - 22v^2w^3 + 9vw^4 - 9w^5) - u^3(9v^5 - 4v^4w + 18v^3w^2 - 4v^2w^3 + 18vw^4 + 9w^5) + u^2(14v^6 - 18v^5w + 18v^4w^2 + 9v^2w^4 + 9vw^5 + 2w^6))] \times [27(u - v)v(u - w)(v - w)w(v^3 + u^2w - u(v^2 + w^2))^2]^{-1}$$

$$d = [-243(u - v)^2v^2(u - w)^2(v - w)^2w^2(v^3 + u^2w - u(v^2 + w^2))^3(u^3w$$

$+v^2(v^2+3vw^2-3w^3)-u(3v^3w+w^3-3vw^3)+u^2v(-3w^2+v(-1+3w)))][(2v^8+2u^6w^2-u^5(4v^2w+9w^3)-u(9v^7+4v^4w^3)+u^4(2v^4+9v^3w+9v^2w^2+14w^4)-u^3(9v^5+5v^4w+18v^3w^2+5v^2w^3+9w^5)+u^2(14v^6+9v^4w^2+9v^3w^3+2w^6))(2u^6w^2+v^5(-3v^2+2v^3-9vw^2+9w^3)-u^5w(4v^2+3w(1+3w))+u^2(14v^6+v^5(3-18w)+9vw^5+w^5(-3+2w)+3v^2w^3(-1+3w)+3v^4w(-1+6w))-u^3(9v^5+v^4(3-4w)+v^2(3-4w)w^2+18vw^4+3w^4(-1+3w)+3v^3w(1+6w))+uv^2(-9v^5+9v^3w^2+v^2(3-22w)w^2-9w^5+3vw^3(1+3w)+v^4(3+9w))+u^4(2v^4+6v^2w+9v^3w+9vw^3+w^3(3+14w)))(2u^6w^2+u^5w(-4v^2+3(1-3w)w)+v^5(3v^2+2v^3-18vw^2+18w^3)+u^2(14v^6-9v^3w^3+18vw^5+w^5(3+2w)+3v^2w^3(1+6w)+3v^4w(1+9w)-3v^5(1+12w))+u^3(-9v^5+3v^3(1-6w)w-36vw^4-3w^4(1+3w)+v^4(3+13w)+v^2w^2(3+13w))+u^4(2v^4+9v^3w+18vw^3-3v^2w(2+3w)+w^3(-3+14w))-uv^2(9v^5+v^4(3-18w)-18v^3w^2+3v(1-6w)w^3+18w^5+v^2w^2(3+40w))))]^{-1}.$

Proof. To classify cubic polynomials with preperiodic points of type 1_3. If $g(y)=ay^3+by+d$ has μ as a rational preperiodic point of type 1_3, then by Theorem 3.2

$$a=[\lambda(-\mu^3+\lambda^2\nu+\mu^2\nu-\nu^3+\lambda\mu(-\mu+\nu))][(\lambda-\nu)(-\mu+\nu)(\lambda+\mu+\nu)]^{-1}$$

$$b=\frac{\lambda-\mu}{((\lambda-\nu)(-\mu+\nu)(\lambda+\mu+\nu)}$$

$$d=\frac{-\lambda^3+\mu^3}{(\lambda-\nu)(-\mu+\nu)(\lambda+\mu+\nu)}$$

we may assume that $g(\xi)=\nu$ without loss of generality, Hence

$$\nu=g(\xi)=a\xi^3+b\xi+d$$

$$0=[-\lambda^2(\mu-\nu)^2-\mu^2\nu^2+\nu^4+\lambda^3(\nu-\xi)+\mu^3\xi-\mu\xi^3-\lambda(\mu^3-2\mu^2\nu+\mu\nu^2+\nu^3-\xi^3)][(\lambda-\nu)(-\mu+\nu)(\lambda+\mu+\nu)]^{-1}$$

Since λ,μ,ν and ξ must be distinct, then Setting $u=\mu-\lambda$, $v=\nu-\lambda$ and $w=\xi-\lambda$, and we assume moreover that $\lambda+\mu+\nu\neq 0$ or $3\lambda+u+v\neq 0$, to make the denominator not equal zero, that gives

$$\lambda=\frac{u^2v^2-v^4-u^3w+uw^3}{3(v^3+u^2w-u(v^2+w^2))}.$$

Moreover, $\mu=\lambda+u$, $\nu=\lambda+v$ and $\xi=\lambda+w$ where a, b and d are obtained from above.

Theorem 3.4 *If $g(y)=ay^3+by+d$ with a,b and $d\in\mathbb{Q}$, Then $g(y)$ has rational preperiodic points of type 2_1 if and only if*

$$a=\frac{u_3}{u(2u_1+u_2+uu_3)}$$

$$b=-\frac{u_1^2+u_2^2+uu_2u_3+u^2u_3^2+u_1(u_2+2uu_3)}{uu_3(2u_1+u_2+uu_3)}$$

$$d=\frac{(u_1+u_2)(uu_3(u_2+uu_3)+u_1(u_2+2uu_3))}{uu_3^2(2u_1+u_2+uu_3)}$$

$u\neq 0, u_3\neq 0, 2u_1+u_2+uu_3\neq 0$, In this case, $q_1=(u_1+uu_3)/u_3$ is a periodic point of type 2_1 and its orbits are $p_1=u_1/u_3$ and $p_2=u_2/u_3$.

Proof. To classify cubic polynomials with preperiodic points of type 2_1. If $g(y)=ay^3+by+d$ has a rational preperiodic point p of type 2_1, then by Theorem 2.2

$$p=\frac{u_1}{u_3}$$

$$a=-\frac{u_3^2}{u_1^2+u_1u_2+u_2^2+u_3u_4}$$

$$b=-\frac{u3u4}{u_1^2+u_1u_2+u_2^2+u_3u_4}$$

$$d=\frac{(u_1+u_2)(u_1^2+u_2^2+u_3u_4)}{u_3(u_1^2+u_1u_2+u_2^2+u_3u_4)}$$

We may assume that $g(q)=p$ without loss of generality, Hence

$$p=g(q)=aq^3+bq+d$$

$$0=[u_2^3+u_2u_3u_4-qu_3^2(q^2u_3+u_4)]\times[u_1^2+u_1u_2+u_2^2+u_3u_4]^{-1}$$

or

$$u_4 = -\frac{(u_2^2 + qu_2u_3 + q^2u_3^2)}{u_3}.$$

Theorem 3.5 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has rational preperiodic points of type 2_2 if and only if*

$$a = [-u^2 - u(v+w) + v(v+w)][u(u-v) \times v(v+w)]^{-1}$$

$$b = [u^6 + u^5(2v+3w) + v^2(v+w)^2(v^2+vw+w^2) - uv(v+w)^2(3v^2+vw+2w^2) + u^4(-2v^2+vw+4w^2) + u^3 \times (-2v^3 - 5v^2w + 3w^3) + u^2(4v^4+6v^3w - vw^3 + w^4)][3u(u-v)v(v+w)(u^2+u(v+w) - v(v+w))]^{-1}$$

$$d = [(2u^3 + uw(v+w) + u^2(2v+3w) - v(2v^2+3vw+w^2))(u^6 + u^5(2v+3w) - uv(v+w)^2(9v^2+vw-4w^2) + v^2(v+w)^2(v^2+vw-2w^2) + u^4(-8v^2-5vw+w^2) - u^3(2v^3+11v^2w+12vw^2+3w^3) + u^2(16v^4+30v^3w+15v^2w^2-vw^3-2w^4))][27u(u-v)v(v+w)(u^2+u(v+w)-v(v+w))^2]^{-1}$$

$u \neq 0, v \neq 0,, w \neq 0$ and $u^2 + u(v+w) - v(v+w) \neq 0$, In this case, q_2 is a periodic point of type 2_2 and its orbits are q_1, p_1 and p_2 where.

$$q_2 = [-u^3 + 2u^2v - v(2v^2+3vw+w^2) + u(3v^2+4vw+w^2)][3(u^2+u(v+w) - v(v+w))]^{-1}$$

$$q_1 = [2u^3 + v^3 - vw^2 + u^2(2v+3w) + u(-3v^2-2vw+w^2)][3(u^2+u(v+w) - v(v+w))]^{-1}$$

$$p_1 = [-u^3 - u^2v + v^3 - vw^2 + uw(v+w)][3(u^2+u(v+w) - v(v+w))]^{-1}$$

$$p_2 = [-u^3 - 2uw(v+w) - u^2(v+3w) + v(v^2+3vw+2w^2)][3(u^2+u(v+w) - v(v+w))]^{-1}$$

Proof. To classify cubic polynomials with preperiodic points of type 2_2. If $g(y) = ay^3 + by + d$ has a rational preperiodic point q_2 of type 2_2, then by Theorem 3.4

$$a = \frac{u_3}{u(2u_1 + u_2 + uu_3)}$$

$$b = \frac{-u_1^2 - u_2^2 - uu_2u_3 - u^2u_3^2 - u_1(u_2 + 2uu_3)}{uu_3(2u_1 + u_2 + uu_3)},$$

$$d = \frac{(u_1+u_2)(uu_3(u_2+uu_3) + u_1(u_2+2uu_3))}{uu_3^2(2u_1+u_2+uu_3)},$$

And q_1 is rational preperiodic point of type 2_1 and p_1 and p_2 are the points in its orbits where

$$q_1 = \frac{uu_3 + u_1}{u_3}$$

$$p_1 = \frac{u_1}{u_3}$$

$$p_2 = \frac{u_2}{u_3}$$

We may assume that $g(q_2) = q_1$ without loss of generality, Hence

$$q_1 = f(q_2) = aq_2^3 + bq_2 + d$$

$$0 = [u_1^2(u_2 - q_2u_3) + u_1(u_2^2 - (q_2 - 2u)u_2u_3 - 2u(q_2+u)u_3^2) + u_3(uu_2^2 + q_2^3u_3^2 - u^3u_3^2 - q_2(u_2^2 + uu_2u_3 + u^2u_3^2))][uu_3^2(2u_1+u_2+uu_3)]^{-1}$$

Since $q_2 \neq p_1$, one sets $q_2 = p_1 + v$ and $v \neq u$, and also setting $u_1 = u_2 + u_3w$ and $w \neq u/u_3$ and v/u_3 that gives

$$0 = 3u_3(u_2+u_3w)[u_3(u^3 + 2uw(v+w) - v(v+w)(v+2w) + u^2(v+3w)) + u_2(3u^2 + 3u(v+w) - 3v(v+w))] \times [3uu_1u_3(2u_1+u_2+uu_3)]^{-1}$$

Since $u_3 \neq 0$ and $u_1 \neq 0$ then $u2 + u3w \neq 0$, then the parameter u_2 is given by

$$u_2 = [-u3(u^3 + 2uw(v+w) + u^2(v+3w) - v(v^2+3vw+2w^2))][3(u^2+uv-v^2+uw-vw)]^{-1}.$$

Moreover $u_1 = u_2 + wu_3, p_1 = u_1/u_3$ and $q_2 = p_1 + w$ and q_1, p_2, a, b and d are obtained from above.

Theorem 3.6 *If $g(y) = ay^3 + by + d$ with a, b and $d \in \mathbb{Q}$, Then $g(y)$ has rational preperiodic points of type 2_3 if $a = 1/(12v^2), b = -19/12$ and $d = v/2$, and $q_3 = v$ is a preperiodic point of type 2_3 and the other points in its orbit are $q_2 = -v, q_1 = 2v, p_1 = -2v$ and $p_2 = 3v$.*

Proof. To classify cubic polynomials with preperiodic points of type 2_3. If $g(y) = ay^3 + by + d$ has a rational preperiodic point q_3 of type 2_2, then by Theorem 3.5

$$a = \frac{-u^2 - u(v+w) + v(v+w)}{u(u-v)v(v+w)}$$

$$b = [u^6 + u^5(2v+3w) + v^2(v+w)^2(v^2 + vw + w^2) - uv(v+w)^2(3v^2 + vw + 2w^2) + u^4(-2v^2 + vw + 4w^2) + u^3(-2v^3 - 5v^2w + 3w^3) + u^2(4v^4 + 6v^3w - vw^3 + w^4)] \times [3u(u-v) \times v(v+w)(u^2 + u(v+w) - v(v+w))]^{-1},$$

$$d = [(2u^3 + uw(v+w) + u^2(2v+3w) - v(2v^2 + 3vw + w^2))(u^6 + u^5(2v+3w) - uv(v+w)^2(9v^2 + vw - 4w^2) + v^2(v+w)^2(v^2 + vw - 2w^2) + u^4(-8v^2 - 5vw + w^2) - u^3(2v^3 + 11v^2w + 12vw^2 + 3w^3) + u^2(16v^4 + 30v^3w + 15v^2w^2 - vw^3 - 2w^4))][27u(u-v)v(v+w)(u^2 + u(v+w) - v(v+w))^2]^{-1},$$

And q_2 is rational preperiodic point of type 2_2 and q_1, p_1 and p_2 are its orbits where

$$q_2 = [-u^3 + 2u^2v - v(2v^2 + 3vw + w^2) + u(3v^2 + 4vw + w^2)][3(u^2 + u(v+w) - v(v+w))]^{-1}$$

$$q_1 = [2u^3 + v^3 - vw^2 + u^2(2v+3w) + u(-3v^2 - 2vw + w^2)][3(u^2 + u(v+w) - v(v+w))]^{-1}$$

$$p_1 = -\frac{u^3 + u^2v - v^3 + vw^2 - uw(v+w)}{3(u^2 + u(v+w) - v(v+w))}$$

$$p_2 = [-u^3 - 2uw(v+w) - u^2(v+3w) + v(v^2 + 3vw + 2w^2)][3(u^2 + u(v+w) - v(v+w))]^{-1}$$

We may assume that $g(q_3) = q_2$ without loss of generality, Hence

$$q_2 = f(q_3) = aq_3^3 + bq_3 + d$$

$$0 = [-27q3^3(u^2 + u(v+w) - v(v+w))^3 + 9u(u-v)v(v+w)(u^2 + u(v+w) - v(v+w))(u^3 - 2u^2v + v(2v^2 + 3vw + w^2) - u(3v^2 + 4vw + w^2)) + (2u^3 + uw(v+w) + u^2(2v+3w) - v(2v^2 + 3vw + w^2))(u^6 + u^5(2v+3w) - uv(v+w)^2(9v^2 + vw - 4w^2) + v^2(v+w)^2(v^2 + vw - 2w^2) + u^4(-8v^2 - 5vw + w^2) - u^3(2v^3 + 11v^2w + 12vw^2 + 3w^3) + u^2(16v^4 + 30v^3w + 15v^2w^2 - vw^3 - 2w^4)) + 9q_3(u^2 + u(v+w) - v(v+w))(u^6 + u^5(2v+3w) + v^2(v+w)^2 \times (v^2 + vw + w^2) - uv(v+w)^2(3v^2 + vw + 2w^2) + u^4(-2v^2 + vw + 4w^2) + u^3(-2v^3 - 5v^2w + 3w^3) + u^2(4v^4 + 6v^3w - vw^3 + w^4))][27u(u-v)v(v+w)(u^2 + u(v+w) - v(v+w))^2]^{-1}$$

Since $q_3 \neq p_1$, one sets $q_3 = p_1 + s$ and $s \neq u, v$, and w and that gives

$$0 = [-u(u-v)v(v+w)^2 - s^3(u^2 + u(v+w) - v(v+w)) + s^2(u^3 + u^2v - v^3 + vw^2 - uw(v+w)) + s(u^3w + uv^2(v+w) - v^2w(v+w) + u^2(-v^2 + w^2))][u(u-v) \times v(v+w)]^{-1}$$

we can see that $u = 4v, w = -5v$, and $s = 3v$ satisfies the last equation, Moreover q_2, q_1, p_1, p_2, a, b and d are obtained from above and $q_3 = p_1 + s$.

Conclusion

In this paper we introduced a Complete parametrization of cubic polynomials that has:

1-Rational periodic points of period 1.

2-Rational periodic points of period 2.

3-Rational periodic points of period 1 and

4-Rational preperiodic points of types $1_1, 1_2, 1_3, 2_1, 2_2$ and 2_3.

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