MANSOURA JOURNAL OF Mathematics
Official Journal of Faculty of Science, Mansoura University, Egypt
E-mail: scimag@mans.edu.eg ISSN: 2974-4946

Classes of harmonic starlike functions defined byRuscheweyh-type q-differential operators

M. A. Mowafy [1], *A. O. Mostafa* [1] *and S. M. Madian* [3]

[1,2] *Dept. of Mathematics, Faculty of Science, Mansoura University, Egypt*

[3] *Basic Science Dept. Higher Institute of Eng. Tec, New Damietta, Egypt).*

*** Correspondence to: (**[1] *mohamed1976224@gmail.com, tel***:010031172805)**

Received:23/7/2023
Accepted:16/10/2023

Abstract :Sufficient and necessary coefficient bounds and other properties are obtained for a class of $M\delta_q^m(\tau,\gamma,\alpha)$, extreme points of closed convex hulls, and distortion theorems are determined for a family of harmonic starlike functions of complex order involving Ruscheweyh-type q-differential operator.

keywords: Harmonic univalent functions, q-calculus, Ruscheweyh-type differential operator and distortion theorems

Introduction

Let Λ be the class of functions:

$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,$ (1.1)

which are analytic in $U=\{z:z\in \mathbb{C}.|z|<1\}$. Also let δ denote the subclass of Λ consisting of univalent functions in U.

For h given by (1.1) and g given by

$g(z)=z+\sum_{k=2}^{\infty}b_k z^k,$ (1.2)

the Hadamard product

(Or convolution) is

$$(h*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k.$$

For $h\in\delta.0<q<1,$ the q-derivative operator ∇_q is given by

(Jackson [7]) and other authors studied q-derivative operator ∇_q such as ([1-5], [10]).

$$\nabla_q h(z)=\begin{cases}\dfrac{h(z)-h(qz)}{(1-q)z} & ,z\neq 0\\ h'(z) & ,z=0\end{cases}$$

that is

$$\nabla_q h(z)=1+\sum_{k=2}^{\infty}[k]_q a_k z^{k-1},$$

where

$$[k]_q=\frac{1-q^k}{1-q},$$

$$[k]_q=0.$$

(Kannas and Rǎducanu [9])

introduced and investigated the Ruscheweyh type q- differential operator

$R_q^m h(z)=h(z)*F_{q,m+1}(z)=z+\sum_{k=2}^{\infty}X_q(k,m)a_k z^k, m>-1,$

(1.3)

where

$$F_{q,m+1}(z)=z+\sum_{k=2}^{\infty}X_q(k,m))\,z^k.$$

and $X_q=\frac{\Gamma_q(k+m)}{(k-1)!\Gamma_q(k+m)}.$ (1.4)

Observe that

$$R_q^0 h(z) = h(z)$$

$$R_q^1 h(z) = z\nabla_q h(z) \ldots$$

$$R_q^m h(z) = \frac{z\nabla_q^m (z^{m-1} h(z))}{m!} = z + \sum_{K=2}^{\infty} X_q(k,m) a_k z^k. \quad (1.5)$$

Let M be the family of harmonic functions $f = h + \bar{g}$ that are orientation preserving and univalent in U where h as in (1.1) and

$$g(z) = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1. \quad (1.6)$$

and $\bar{M} \subset M$ consisting of $f = h + \bar{g}$ where

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, \ a_k \geq 0 \ and \ b_k \geq 0. \quad (1.7)$$

For $f \in M . R_q^m g(z)$, be defined by

$$R_q^0 g(z) = g(z)$$

$$R_q^1 h(z) = z\nabla_q g(z) \ldots$$

$$R_q^m g(z) = \frac{z\nabla_q^m (z^{m-1} g(z))}{m!} = z + \sum_{k=1}^{\infty} X_q(k,m) b_k z^k. \quad (1.8)$$

Recently, (Jahangiri [8]) applied q-difference operators to classes of harmonic functions and obtained coefficient bounds for such functions. Motivated by ([8] and [9]) , we define the class $M_q^m(\alpha)$ of Ruscheweyh-type q- calculus harmonic functions $M_q^m(\alpha)$ consisting of $f \in M$ satisfying

$$Re\left(\frac{R_q^{m+1} f(z)}{R_q^m f(z)}\right) \geq \alpha; 0 \leq \alpha \leq 1,$$

where $R_q^m h(z)$ and $R_q^m g(z)$ are, respectively, given by (1.5) , (1.8) and

$$R_q^m f(z) = R_q^m h(z) + (-1)^m \overline{R_q^m g(z)}, m > -1. \quad (1.9)$$

The subfamily $\bar{M}_q^m(\alpha) \subset M_q^m(\alpha)$ consists of harmonic functions $f = h + \bar{g}_m$ for which

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, g_m(z) = (-1)^m \sum_{k=1}^{\infty} b_k z^k, \ a_k \geq 0 \ and \ b_k \geq 0. \quad (1.10)$$

Definition 1:

For non-zero complex number τ *with* $|\tau| \leq 1 . \gamma \in \mathbb{R}$ *and* $0 \leq \alpha \leq 1$ let $M\delta_q^m(\tau.\gamma.\alpha)$ *be the subclass of* $f \in M$ *satisfying*

$$Re\left\{1 + \frac{1}{\tau}\left[\left(1 + e^{i\gamma}\right)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - \left(1 + e^{i\gamma}\right)\right]\right\} > \alpha \quad (1.11)$$

and

$$\overline{M\delta_q^m}(\tau,\gamma,\alpha) \subset M\delta_q^m(\tau,\gamma,\alpha) \cap \bar{M}.$$

Note that:

$(i) M\delta_q^m(1,\gamma,\alpha) \equiv M\delta_q^m(\gamma,\alpha)$ (see [11]) :

$$Re\left[\left(1 + e^{i\gamma}\right)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - e^{i\gamma}\right] > \alpha.$$

$(ii) M\delta_q^m(\tau, 0, \alpha) \equiv M\delta_q^m(\tau, \alpha)$ (see [6] Definition1) :

$$Re\left[1 + \frac{2}{\tau}\left(\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - 1\right)\right] > \alpha.$$

2 M ain section

Unless otherwise mentioned we shall assume that $|\tau| \leq 1, \gamma \in \mathbb{R} . 0 \leq \alpha \leq 1, m > -1, 0 < q < 1$ and $X_q(k,m)$ is given by (1.4).

Theorem 2.1. *For* $f \in M$. *If*

$$\sum_{k=1}^{\infty}\left[\begin{array}{c}\frac{\frac{2[k+m]_q}{[1+m]_q} + 2 - (1-\alpha)|\tau|}{(1-\alpha)|\tau|} x_q(k,m)|a_k| \\ + \frac{\frac{2[k+m]_q}{[1+m]_q} - 2 + (1-\alpha)|\tau|}{(1-\alpha)|\tau|} x_q(k,m)|b_k|\end{array}\right] \leq 2 \quad (2.1)$$

then f *is orientation-preserving harmonic univalent in U and* $f \in M\delta_q^m(\tau,\gamma,\alpha)$.

Proof.

Let (2.1) hold, first we prove that f orientation-preserving in U it is sufficient to show that $\left|R_q^m h(z)\right| \geq \left|R_q^m g(z)\right|$.

$$\left|R_q^m h(z)\right| \geq 1-\sum_{k=2}^{\infty} X_q(k,m+1)|a_k|r^{k-1}$$

$$> 1-\sum_{k=2}^{\infty} X_q(k,m+1)|a_k|.$$

$$\geq 1 - \sum_{k=2}^{\infty}\left\{\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|a_k|\right\}$$

$$\geq \sum_{k=1}^{\infty}\left\{\frac{\frac{2[k+m]_q}{[1+m]_q}+2-(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|b_k|\right\}$$

$$\geq \sum_{k=1}^{\infty} X_q(k,m+1)|b_k|$$

$$\geq \sum_{k=1}^{\infty} X_q(k,m+1)|b_k|r^{k-1} \geq \left|R_q^m g(z)\right|$$

If $z_1 \neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1-\left|\frac{\sum_{k=1}^{\infty} b_k({z_1}^k-{z_2}^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k({z_1}^k-{z_2}^k)}\right|$$

$$> 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|}$$

$$\geq 1-\frac{\frac{\frac{2[k+m]_q}{[1+m]_q}+2-(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|b_k|}{\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|a_k|} \geq 0$$

This gives the univalence of f.

Finally, we prove that $f \in M\delta_q^m(\tau,\gamma,\alpha)$. Using the fact that,

$$Re(\omega(z)) \geq \alpha \Leftrightarrow |1-\alpha+\omega| \geq |1+\alpha-\omega|,$$

we may show that

$$\left|\left[2\tau-\alpha\tau-\left(1+e^{i\gamma}\right)\right]\left[R_q^m h(z)+(-1)^m\overline{R_q^m g(z)}\right]\left(1+e^{i\gamma}\right)\left[R_q^{m+1}h(z)-(-1)^m\overline{R_q^{m+1}g(z)}\right]\right|-\left|\left(1+\alpha\tau+e^{i\gamma}\right)\left[R_q^m h(z)+(-1)^m\overline{R_q^m g(z)}\right]-\left(1+e^{i\gamma}\right)\left[R_q^{m+1}h(z)-(-1)^m\overline{R_q^{m+1}g(z)}\right]\right| \geq 0.$$

$$\left|\left[2\tau-\alpha\tau-\left(1+e^{i\gamma}\right)\right]\left[z+\sum_{k=2}^{\infty} X_q(k,m)a_k z^k+(-1)^m\sum_{k=1}^{\infty} X_q(k,m)b_k \bar{z}^k\right]+\left(1+e^{i\gamma}\right)\left[z+\sum_{k=2}^{\infty} X_q(k,m+1)a_k z^k-(-1)^m\sum_{k=1}^{\infty} X_q(k,m+1)b_k \bar{z}^k\right]\right|-\left|\left(1+\alpha\tau+e^{i\gamma}\right)\left[z+\sum_{k=2}^{\infty} X_q(k,m+1)a_k z^k+(-1)^m\sum_{k=1}^{\infty} X_q(k,m+1)b_k \bar{z}^k\right]-\left(1+e^{i\gamma}\right)\left[z+\sum_{k=2}^{\infty} X_q(k,m)a_k z^k-(-1)^m\sum_{k=1}^{\infty} X_q(k,m)b_k \bar{z}^k\right]\right| \geq (2-\alpha)|\tau||z|$$

$$-\sum_{k=2}^{\infty}\left|(2-\alpha)\tau+\left(1+e^{i\gamma}\right)\left(\frac{[k+m]_q}{[1+m]_q}+1\right)(2-\alpha)\tau\right| X_q(k.m)|a_k||z^k| - \sum_{k=1}^{\infty}\left|\left(1+e^{i\gamma}\right)\left(\frac{[k+m]_q}{[1+m]_q}+1\right)-(2-\alpha)\tau\right| X_q(k,m)|b_k||z^k|$$

$$-\alpha|\tau||z|-\sum_{k=2}^{\infty}\left|\left(1+e^{i\gamma}\right)\left(\frac{[k+m]_q}{[1+m]_q}-1\right)-\alpha\tau\right| X_q(k.m)|a_k||z^k|$$

$$-\alpha|\tau||z|-\sum_{k=2}^{\infty}\left|\left(1+e^{i\gamma}\right)\left(\frac{[k+m]_q}{[1+m]_q}-1\right)-\alpha\tau\right| X_q(k.m)|a_k||z^k|$$

$$-\sum_{k=1}^{\infty}\left|\left(1+e^{i\gamma}\right)\left(\frac{[k+m]_q}{[1+m]_q}+1\right)+\alpha\tau\right| X_q(k,m)|b_k||z^k|.$$

$\geq 2(1-\alpha)|\tau||z|\{2$

$$-\sum_{k=1}^{\infty}[\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|a_k|$$

$$+\sum_{k=1}^{\infty}[\frac{\frac{2[k+m]_q}{[1+m]_q}+2-(1-\alpha)|\tau|}{(1-\alpha)|\tau|}X_q(k,m)|b_k|.$$

$$\geq 0$$

This completes the proof.

The function

$$f(z)=\sum_{k=2}^{\infty}\frac{(1-\alpha)|\tau|}{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}x_{k^{z^k}}+$$

$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)|\tau|}{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}\bar{y}_{k^{\bar{z}^k}}$$

Where $\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$,which give the charppness for (2.1).

Theorem 2.2. Let $f_m=h+\bar{g}_m$ given by (1.10), $a_1=1$ Then f_m is orientation-preserving harmonic univalent in U and $f_m\in M_q^m$.if and only if

$$\sum_{k=1}^{\infty}\left[\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}{(1-\alpha)|\tau|}x_q(k.m)a_k+\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)|\tau|}{(1-\alpha)|\tau|}x_q(k.m)b_k\right]\leq 2.\quad(2.2)$$

Proof. Since $\bar{M}_q^m(\tau,\gamma,\alpha)\subset M_q^m(\tau,\gamma,\alpha)$,

then first part hold from theorem 2.1. The second part, we will show that if (2.2) does not hold then $f_m\notin \bar{M}_q^m(\tau,\gamma,\alpha)$.

For $f_m\notin \bar{M}_q^m(\tau,\gamma,\alpha)$.

$$\mathrm{Re}\left\{1+\frac{1}{\tau}\left[\left(1+e^{i\gamma}\right)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-\left(1+e^{i\gamma}\right)\right]\right\}\geq\alpha.$$

Or equivalently

$$\mathrm{Re}\left\{\frac{(1-\propto)\tau z-\sum_{k=2}^{\infty}\left[(1-\alpha)\tau+\left(\frac{[k+m]_a}{[1-m]_q}-1\right)\left(e^{i\gamma}+1\right)\right]z_q(k.m)a^kz^k}{\tau\left[z-\sum_{k=2}^{\infty}z_q(k.m)a^{k2^k}+(-1)^{2m}\sum_{k=1}^{\infty}z_q(k.m)a^k\bar{z}^k\right]}\right\}$$

$$\mathrm{Re}\left\{\frac{(-1)^{2n}\sum_{k=1}^{\infty}\left[\left(\frac{[k+m]_q}{[1-m]_q}-1\right)\left(e^{i\gamma}+1\right)-(1-\alpha)\tau\right]\bar{\tau}X_q(k,m)a^kz^k}{\tau\left[z-\sum_{k=2}^{\infty}X_q(k,m)a^{k2^k}+(-1)^{2m}\sum_{k=1}^{\infty}X_q(k,m)a^k\bar{z}^k\right]}\right\}$$

$$=\mathrm{Re}\left\{\frac{(1-\alpha)|\tau|^2-\sum_{k=2}^{\infty}\left[(1-\alpha)\tau+\left(\frac{[k+m]_a}{[1-m]_q}-1\right)\left(e^{i\gamma}+1\right)\right]X_q(k,m)a^k}{|\tau|^2\left[1-\sum_{k=2}^{\infty}X_q(k,m)a^kz^{k-1}+\frac{\bar{z}}{z}\sum_{k=1}^{\infty}X_q(k,m)a^k\bar{z}^{k-1}\right]}\right\}$$

$$-\mathrm{Re}\left\{\frac{\frac{\bar{z}}{z}\sum_{k=2}^{\infty}\left[(1-\alpha)\tau+\left(\frac{[k+m]_a}{[1-m]_q}-1\right)\left(e^{i\gamma}+1\right)\right]X_q(k,m)a^kz^{k-1}}{|\tau|^2\left[1-\sum_{k=2}^{\infty}X_q(k,m)a^kz^{k-1}+\frac{\bar{z}}{z}\sum_{k=1}^{\infty}X_q(k,m)a^k\bar{z}^{k-1}\right]}\right\}\geq 0$$

The above condition must hold $\forall\,\gamma,\lfloor z\rfloor=r<1\ and\ 0<|\tau|<1. for\ \gamma=0\ and\ |\tau|=\tau\ let\ 0<z=r<1. then\ (2.3) becomes$

$$\frac{(1-\alpha)|\tau|^2-\sum_{k=2}^{\infty}\left[\left(\frac{2[k+m]_q}{[1-m]_q}-2\right)(1-\alpha)\tau\right]|\tau|X_q(k,m)a^kz^{k-1}}{|\tau|^2\left[1-\sum_{k=2}^{\infty}X_q(k,m)a^{k2^{k-1}}+\sum_{k=1}^{\infty}X_q(k,m)a^k\bar{z}^{k-1}\right]}$$

$$-\frac{\sum_{k=1}^{\infty}\left[\left(\frac{2[k+m]_q}{[1-m]_q}+2\right)(1-\alpha)\tau\right]|\tau|X_q(k.m)a^kz^{k-1}}{|\tau|^2\left[1-\sum_{k=2}^{\infty}X_q(k,m)a^kz^{k-1}+\sum_{k=1}^{\infty}X_q(k,m)a^k\bar{z}^{k-1}\right]}\geq 0$$

Observer that the numerator in)2.3) is negative if (2.2) does not holt. Thus $\exists$ appoint$z_0=r_0$ in (0.1) for which (2.4) is negative, which contradicts (1.11) for $f_m\epsilon\bar{M}\delta_q^m(\tau,\gamma,\alpha)$. hence the proof is completed.

Theorem 2.3. Let f_m be given by (1.10), then it is orientation preserving harmonic univalent in U and $f_m\epsilon\bar{M}\delta_q^m(\tau.\gamma.\alpha)$ if and only

$$\text{if}\sum_{k=1}^{\infty}\left[\frac{\frac{2[k+m]_q}{[1+m]_q}-2+(1-\alpha)}{(1-\alpha)}X_q(k,m)_{a_k}+\frac{\frac{2[k+m]_q}{[1+m]_q}+2-(1-\alpha)}{(1-\alpha)}X_q(k,m)b_k\right]\leq 2.$$

Theorem 2.4. let fm be given by (1.10), then $f_m\epsilon clco\bar{M}\delta_q^m(\tau.\gamma.\alpha)$ if and only if

$$f_m=\sum_{k=1}^{\infty}\left(x_kh_k+Y_kg_{m_k}\right),\quad(2.5)$$

Where

$h_1(z)=z,\ h_k(z)=z-\sum_{k=2}^{\infty}\frac{X_q(k.m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}z^k,k=2,3,\dots;$

$,g_{m_k}(z)=z+(-1)^m\frac{(1-\alpha)|\tau|}{x_q(k.m)\left[\frac{2[k+m]_q}{[1+m]_q}+2-(1-\alpha)[\tau]\right]}\bar{z}^k$

,k=1,2,…;

$\sum_{k=1}^{\infty}(X_k+Y_k)=1,\ X_k\ge 0$ and $Y_k\ge 0$

In particular , the extreme points of $clco\bar{M}\delta_q^m(\tau,\gamma,\alpha)$ are $\{h_k\}$ and $\{g_{m_k}\}$

Proof. Assume that f_m as in (2.5),then

$f_m=\sum_{k=1}^{\infty}\left(X_k h_k+Y_k g_{m_k}\right)=\sum_{k=1}^{\infty}(X_k+Y_k)z$

$-\sum_{k=2}^{\infty}\frac{(1-\alpha)|\tau|}{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}X_{k^{z^k}}$

$+(-1)^m\sum_{k=2}^{\infty}\frac{(1-\alpha)|\tau|}{x_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}\bar{z}^k$

Since

$\sum_{k=2}^{\infty}\frac{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}$

$$\left\{\frac{(1-\alpha)|\tau|}{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}\right\}X_k$$

$$+\sum_{k=2}^{\infty}\frac{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}$$

$$\left\{\frac{(1-\alpha)|\tau|}{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}\right\}Y_k$$

$=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\le 1.$

Thus $f_m\epsilon clco\bar{M}\delta_q^m(\tau,\gamma,\alpha)$.

Conversely, suppose that $f_m\epsilon clco\bar{M}\delta_q^m(\tau,\gamma,\alpha)$.

Set

$X_k=\frac{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|a_k|,k=$ 2,3, ...,

and

$Y_k=\frac{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|b_k|,k=$ 1,2, ...,

Where $\sum_{k=2}^{\infty}(X_k+Y_k)=1$. Then

$f_m=$
$z-\sum_{k=2}^{\infty}a_k z^k+(-1)^m\sum_{k=2}^{\infty}b_k\bar{z}^k=z-\sum_{k=2}^{\infty}\frac{(1-\alpha)|\tau|}{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}X_k z^k+$

$(-1)^m\sum_{k=1}^{\infty}\frac{(1-\alpha)|\tau|}{X_q(k,m)\left[\frac{2[k+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}Y_k\bar{z}^k=$
$z+\sum_{k=2}^{\infty}[X_k(h_k(z)-z)]$

$+\sum_{k=1}^{\infty}[Y_k(gm_k(z)-z)]$

$=\sum_{k=1}^{\infty}[X_k(h_k(z)+Y_k(gm_k)]$

As required.

Theorem 2.5. Let $f_m\epsilon\bar{M}\delta_q^m(\tau,\gamma,\alpha)$

where $|z|=r<1.then$

$|f_m|\le$
$(1+\tau_1)r+\left\{\begin{array}{l}\frac{(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}-\\ \frac{4-(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}|\tau_1|\end{array}\right\}r^2,$ (2.6)

and

$|f_m|\le$
$(1-\tau_1)r-\left\{\begin{array}{l}\frac{(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}-\\ \frac{4-(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}|\tau_1|\end{array}\right\}r^2.$ (2.7)

$|f_m|\le(1+\tau_1)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\le(1+\tau_1)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)=(1+\tau_1)r+\frac{(1-\alpha)|\tau|r^2}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}\times$
$\sum_{k=2}^{\infty}X_q(2,m)\left[\frac{\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|a_k|+\frac{\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|b_k|\right]\le(1+\tau_1)r+$
$\frac{(1-\alpha)|\tau|r^2}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}\times$
$\sum_{k=2}^{\infty}X_q(2,m)\left[\frac{\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|a_k|+\frac{\left[\frac{2[2+m]_q}{(1+m)_q}+2-(1-\alpha)|\tau|\right]}{(1-\alpha)|\tau|}|b_k|\right]\le(1+\tau_1)r+$
$\frac{(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}\times$

$$r^2\left[1-|\tau_1|\frac{4-(1-\alpha)|\tau|}{(1-\alpha)|\tau|}\right]$$

$\le$

$$(1+\tau_1)r+\left\{\begin{array}{l}\frac{(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}-\\ \frac{4-(1-\alpha)|\tau|}{X_q(2,m)\left[\frac{2[2+m]_q}{(1+m)_q}-2+(1-\alpha)|\tau|\right]}|\tau_1|\end{array}\right\}r^2.$$

3 Conclusion

In this paper we determined coefficient bounds and other properties are obtained for a class of $M\delta_q^m(\tau,\gamma,\alpha)$ extreme points of closed convex hulls, and distortion theorems are determined for a family of harmonic starlike functions of complex order involving Ruscheweyh-type q-differential operator

4 Acknowledgement

The authors wish to thank Prof. Dr. M. K. Aouf for his kind encouragement and help in the preparation of this paper.

References

1 M. H. Abu-Risha, M. H. Annaby, M. E. H. Ismail and S. Mansour, Linear q-(2007), difference equations, Z. Anal. Anwend**. 26** 481-494.

2 M. H. Annaby and Z. S. Mansour, q-(2012) Fractional Calculus and Equations, Lecture notes in Math., vol. 2056. Springer, Berlin

3 M. K. Aouf and A. O. Mostafa, (2020), Subordination results for analytic functions associated with fractional $q-$ calculus operators with complex order, Afr. Mat., 31 1387-1396.

4 M. K. Aouf and A. O. Mostafa, (2020), some subordinating results for classes of functions defined by Sălăgean type $q-$ derivative operator, Filomat., 34 no. **7**, 2283-2292.

5 M. K. Aouf, A. O. Mostafa and R. E. (2021),functions with varying arguments associated with $q-$ difference operator, Afrika Math., **32** 621-630

6 T. Bulboaca, M. A. Nasr and G. F. Sălăgean, A (1992), generalization of some classes of starlike functions of complex order, Mathematica (Cluj), 34, no. 57, 113-118.

7 F. H. Jackson, On q-functions and a certain (1908), difference operator, Trans. R. Soc. Edinb. 46 64-72.

8 J. M. Jahangiri, (2018), Harmonic univalent functions defined by q-calculus operators, Inter. *J.Math. Anal. Appl.* 5 no. 2, 39-43.

9 S. Kanas, D. Răducanu, (2014) , some subclass of analytic functions related to conic domains, Math. Slovaca 64 no. 5, 1183-1196.

10 A. O. Mostafa and Z. M. Saleh, (2001) On a class of uniformly analytic functions with q-analogue, *Int. J. Open Problems Complex Analysis*, 13 no. 2, 1-13.

11 T. Rosy, B. A. Stephen, K. G. Subramanian and J. M. Jagangiri, Goodman-Rønning (2001), type harmonic univalent functions, *Kyung pook Math. J.* 41 45-54.