Menofia University Faculty of Engineering Basic Engineering Sci. Department Academic Year : 2019-2020 Date : 10 / 08 / 2020	Subject: Introduction in Partial Differential Equations Code: BES 507 Time Allowed : 3 hours Year : Master Total Marks: 100 Marks

Answer all the following questions: (الامتحان في ثلاث صفحات)

Question 1 — ٢٠ marks

Method of characteristics. Solve the following initial-boundary value problem:

$$u_t + e^{-x}u_x = e^x; \qquad x > 0, t > 0$$
$$u(x,0) = f(x); \qquad x > 0$$
$$u(0,t) = g(t); \qquad t > 0$$

You will need to separate the domain into two regions, and be sure to identify the boundary between the two regions.

Question 2 — ٢٠ marks

Heat Equation.

(a) Use energy methods to show the following initial boundary value problem has at most one solution:

$$u_t(x,t) = f(t)\,\Delta\, u(x,t); \qquad f(t) > 0, x \in \Omega$$
$$u(x,0) = g(x); \qquad x \in \Omega$$
$$u(x,t) = h(x,t); \qquad x \in \partial\Omega, t > 0$$

Assume $f(t) \in C^\infty[0,\infty]$, $g(x) \in C^\infty(\partial\Omega)$, and $h(x,t) \in C^\infty(\partial\Omega \times [0,\infty])$ are all integrable functions and the domain Ω is simply connected.

(b) Find the solution to the heat equation on the n-dimensional half space:

$$u_t(x,t) = k\Delta u(x,t); \qquad x \in \Omega \equiv \{x \in \mathbb{R}^n | x_n > 0\}, t > 0$$
$$u(x,0) = f(x); \qquad x \in \Omega$$
$$\frac{\partial u}{\partial n}(x,t) = 0; \qquad x \in \partial\Omega$$
$$\lim_{|x|\to\infty} u(x,t) = 0; \qquad t > 0$$

Where $k > 0$ is a positive scalar, $f(x)$ is continuous and L^2 integerable on $\mathbb{R}^n$, and n is the unit normal to the boundary ∂Ω.

Question 3 **٢٠ marks**

Wave Equation. Consider the equation:

$$u_{tt}(x,t) = c^2 \Delta u(x,t),$$
$$u(x,0) = f(x); \qquad x \in \mathbb{R}^3,$$
$$u_t(x,0) = g(x); \qquad x \in \mathbb{R}^3,$$

where $c > 0$ is a positive scalar, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are rapidly decaying, C^{∞}, and L^2 integrable functions.

(a) Find the equation the average of u:

$$\overline{u}(\boldsymbol{r},t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(x,t)\, \sin\varphi d\varphi\, d\theta$$

satisfies where $\boldsymbol{x} = (x, y, z)$ and $x = r\cos\theta\sin\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi\ \theta$, such that θ and φ are angles in spherical coordinates.

(b) Assume spherical symmetry of initial conditions ($f = f(r)$, $g = g(r)$) and the solution $u = u(r, t)$, where $r \equiv |x|$, and write the initial / boundary value problem for the radially symmetric function $v(r, t) = ru(r, t)$.

(c) Find the solution $v(r, t)$ and hence the solution $u(r, t)$.

Question 4 **2٠ marks**

Green's Functions.

(a) Consider Poisson's equation on the tilted half plane

$$\Delta u = f(x); \qquad x \in \Omega \equiv \{x \in \mathbb{R}^2 | x_1 + x_2 > 0\}, \qquad (1)$$
$$u(x) = g(x); \qquad x \in \partial\Omega$$

Write the associated Green's function $G_H(x, y)$ using the method of images, and verify its corresponding boundary value problem.

(b) Consider Poisson's equation on the tilted half disc:

$$\Delta u = f(x); \qquad x \in \Omega \equiv \{x \in \mathbb{R}^2 | x_1 + x_2 > 0\ \&|x| < 1\}, \qquad (2)$$
$$u(x) = 0; \qquad x \in \partial\Omega$$

Determine the associated Green's function $G_S(x, y)$ and show it satisfies the needed boundary conditions. Then, write the solution to Eq. (2) in terms of this Green's function, and show it satisfies $u(x) = 0$ on $x \in \partial\Omega$.