Sci. J. Fac. Sci., Menufiya Univ. Vol. XIII (1999) (147 - 162)

All -INTEGER INTEGER SEPARABLE NON - LINEAR PROGRAMS

M. EL-KAFRAWY, M. PANAGA, Z. MO'MIN-KHAN
Faculty of Girls Jeddah, K. S. A.

Abstract

Separable non-linear programming has been treated in non-linear programming materials (see the definition of separable nonlinear rogramming). This paper concerns with a contributed treatment to an integer separable non-linear programming problem, substantiated by two examples to prove our theory. Where as the separable programming approach is at least competitive and probably superior for solving any convex separable program, it should be used on a non-convex problem as well using high speed computers.

1 . **Integer Separable Programming (objective function)**

Def : Functions that can be broken into single variable components as in the form

$$f(X)=\sum_{j=1}^{n} f_j(x_j) \quad , \quad X=(x_1, x_2, \ldots, x_n) \qquad (1.1)$$

Where each of the f_j is a continuous function of a single variable x_j , are said to be separable. For example, any linear function

$$f(X)=\sum_{j=1}^{n} c_j x_j \qquad (1.2)$$

is obviously separable , with each of the component functions being $f_j(x_j) = c_j x_j$. So is any quadratic form that lacks cross-product terms :

$$f(X)=\sum_{j=1}^{n} c_j x_j^2 \qquad (1.3)$$

Optimisation problems with non-linear separable objective functions are commonly found in life . For example , any objective that calls for minimising the total variance of a number of independent random variables (such as investments) would posses the separability property, as would the objective of minimizing raw - material costs at a plant that orders each raw material from a different supplier .

On the other hand, some special forms of objective functions can be made separable by certain transformations of variables . The simplest and most useful transform is applicable to the non-separated term $x_i\, x_j$ defining

$$\begin{aligned} y_i &= (1/2)(x_j + x_j) \\ y_j &= (1/2)(x_j - x_j) \end{aligned} \qquad (1.4)$$

then , it follows that

$$x_j\, x_j = y_i^2 - y_j^2$$

Thus , the term $x_j\, x_j$, in the objective function is replaced by the separable expression y_i^2 - y_j^2 and the two linear equations (1.4) are added to the set of constraints.

Now, consider the separable programming problem

$$\left.\begin{aligned} \text{Max } Z &= \sum_{j=1}^{n} f_j(x_j) \\ \text{subject to } & AX = b \\ \text{and } & X \geq 0 \end{aligned}\right\} \qquad (1.5)$$

where X= ($x_1, \ldots$ x_n) , A is $m \times n$, b is $m \times 1$ and the f_j are continuous functions . The linear constraints are in standard form

. We begin by approximating each of the functions $f_j(x_j)$ as closely as is desired by a piecewise linear function $\hat{f}_j(x_j)$
This is done by determining a lower bound $\underline{x}_j$ and upper bound $\overline{x}_j$ on the value of x_j ; choosing r_j +1 break points (or values of x_j) , denoted $x_{j0}, x_{j1}, \ldots, x_{jr_j}$ where

$$x_{j0} = \underline{x}_j < x_{j1} < x_{j2} < \ldots < x_{jr_j} = \overline{x}_j$$

and computing for each of these values the ordinate

$$f_{jk} = f_j(x_{jk}) \quad , k = 0,1,\ldots,r_j$$

The function $\hat{f}_j(x_j)$ is then the piecewise linear curve that is produced by joining the points

$$(x_{j0}, f_{j0}),\ (x_{j1}, f_{j1}), \ldots, (x_{jr_j}, f_{jr_j})$$

with r_j successive straight - line segments (x_j may be zero).

Algebraically, any general point x_j in the interval

$x_{jk} \le x_j \le x_{j,k+1}$ can be expressed as a unique convex

combination of the two end points:

$$x_j = \lambda_{jk} x_{jk} + \lambda_{j,k+1} x_{j,k+1} \qquad (1.6)$$

where

$$\lambda_{jk} + \lambda_{j,k+1} = 1, \text{and } \lambda_{jk}, \lambda_{j,k+1} \ge 0 \qquad (1.7)$$

The approximated objective value of x_j is then

$$\hat{f}_j(x_j) = \lambda_{jk} f_{jk} + \lambda_{j,k+1} f_{j,k+1} \qquad (1.8)$$

To represent the piecewise linear function $\hat{f}_j$, it is necessary to use logical restrictions in addition to algebraic relations. Any x_j. in the entire range $\underline{x}_j \le x_j \le \overline{x}_j$, together with its approximate objective value , can be expressed uniquely in terms of the variables $\lambda_{j0}, \lambda_{j1},\ldots,\lambda_{jr_j}$ as follows :

$$x_j = \sum_{k=0}^{r_j} \lambda_{jk} x_{jk} \quad (1.9)$$

$$\& \ \hat{f}_j(x_j) = \sum_{k=0}^{r_j} \lambda_{jk} f_{jk} \quad (1.10)$$

$$\text{where } \sum_{k=0}^{r_j} \lambda_{jk} = 1 \text{ and } \lambda_{jk} \ge 0, k = 0,1,\ldots,r_j \quad (1.11)$$

provided it is also required that

(i) At most two of the λ_{jk} can be positive , and

(ii) If two are positive they must be adjacent (i.e. if λ_{js} and λ_{jt} are positive , then either $t = s+1$ or $s = t + 1$)

Now, the approximating problem is constructed by choosing points that define a piecewise linear approximation for each $f_j(x_j)$ and then making substitutions of the form (1.9) - (1.11) for each variable x_j The i^{th} constraint

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

becomes

$$\sum_{j=1}^{n} \sum_{k=0}^{r_j} a_{ijk} \lambda_{jk} = b_i \quad (1.12)$$

where $a_{ijk} = a_{ij} x_{jk}$ and the problem becomes the approximating problem :

$$\left.\begin{array}{lll}\text{Max } Z=\sum_{j=1}^{n} \hat{f}_j(x_j)\equiv \sum_{j=1}^{n}\sum_{k=0}^{r_j} f_{jk}\lambda_{jk} & & \\ \text{s.t.} \quad \sum_{j=1}^{n}\sum_{k=0}^{r_j} a_{ijk}\,\lambda_{jk}=b_i\,, & & i=1,\ldots,m \\ \quad \sum_{k=0}^{r_j}\lambda_{jk}=1\,, & & j=1,\ldots,n \\ \text{and} \quad \lambda_{jk}\geq 0 \quad \forall j\&k & & \\ \text{and to restrictions (i) and (ii) } \forall j\,, & & j=1,\ldots,n \end{array}\right\}(1.13)$$

Which is identical to the original separable program

$$\left.\begin{array}{ll}\text{Max } Z=\sum_{j=1}^{n}\hat{f}_j(x_j) & \\ \text{s.t.} \quad AX=b & \\ \text{and} \quad \underline{x}_j\leq x_j\leq \overline{x}_j\,, & j=1,\ldots,n\end{array}\right\}(1.14)$$

and nearly identical to the original separable program(I.5), such that, $\underline{x}_j$ and $\overline{x}_j$ agree with the feasible region. The closer the approximations $\hat{f}_j$ are to the given functions f_j, the closer (1.13) is to (1.5). Notice that, in forming the approximating problem (1. 13) there is no need to construct a piecewise linear approximation to any function $f_j(x_j)$ that is already linear.

Theorem 1.1 If the piecewise linear functions $\hat{f}_j$ are integers, then the optimum x_j^* are integers.

Proof: From the relation (1.9):

$$x_j = \sum_{k=0}^{r_j} \lambda_{jk} \ x_{jk} \qquad (1.9)$$

and since each λ_{jk} should be zero or one , then x_j^* must be integers

2. **Integer Separable Programming (non-linear constrained)**

To extend and make the separable programming approach described in article 1. applicable to non-linearly constrained problems , we use Miller's [11] method . For the purpose of integrally , we use Theorem 1. 1 . The procedure involved is essentially the same as for linearly constrained problems , except for that piecewise linear approximations which must be constructed for the constraint functions as well as for the objective function .

Thus every non-linear function appearing in the problem must be separable, in the sense defined in article 1. Let the general formulation of the convex separable program is:

$$\left.\begin{array}{ll} \text{Max}\, Z = \sum_{j=1}^{n} f_j\left(x_j\right) & \\ \text{s.t.} \quad \sum_{j=1}^{n} g_{ij}\left(x_j\right) \leq b_i & , i=1,\ldots,u, \\ \sum_{j=1}^{n} g_{ij}\left(x_j\right) \equiv \sum_{j=1}^{n} a_{ij}\, x_j = b_i & , i=u+1,\ldots, v, \\ \& \quad \sum_{j=1}^{n} g_{ij}\left(x_j\right) \geq b_i & , i=v+1,\ldots,m. \end{array}\right\} \quad (2.1)$$

where the functions f_j are concave , the functions g_{ij} , $i = 1 , \ldots , \mathbf{u}$, all j , are convex . the functions g_{ij} $i = v+1 ,\ldots , m$, all j , are concave ; and the a_{ij} and b_i are real numbers. It is not necessary to formulate the problem in terms of non-negative variables .

2.1 The Algorithm

To solve the problem (2.1) , we begin by forming an approximating problem (the same procedure as in a article 1) .For each j , j = 1,...,n, let $\underline{x}_j$ and $\overline{x}_j$ be a lower and upper bound on the value of x_j , and choose a set of $r_j + 1$ integer break points x_{jk} , k = 0 , 1,... , r_j ,satisfying

$$x_{j0} = \underline{x}_j < x_{j1} < x_{j2} < \ldots < x_{jr_j} = \overline{x}_j$$

For each of these values compute the ordinates

$$f_{jk} = f_j(x_{jk}) \tag{2.2}$$

and

$$g_{ijk} = \begin{cases} g_{ij}(x_{jk}), & i = 1,\ldots,u \\ a_{ij}\, x_{jk}, & i = u+1,\ldots,v \\ g_{ij}(x_{jk}), & i = v+1,\ldots,m \end{cases} \tag{2.3}$$

The ordinates f_{jk} and g_{ijk} define piecewise linear functions $\hat{f}_j(x_j)$ and $\hat{g}_{ij}(x_j)$ that can be taken as approximations to the original functions f_j and g_{ij} .

Let a new set of variables $\lambda_{j0}, \lambda_{j1}, \ldots, \lambda_{jr_j}$ be defined for each x_j , j = 1,...,n.

Then, making substitutions of the form (1.9) through (1. 11) for each variable x_j , including a substitution

$$\hat{g}_{ij}(x_j) = \sum_{k=0}^{r_j} \lambda_{jk} g_{ijk}$$

for each of the constraint functions $\hat{g}_{ij}$, $i = 1, \dots, m$, we get the following approximating problem in the variables λ_{jk}

$$\left.\begin{aligned}
&\text{Max } Z = \sum_{j=1}^{n} \hat{f}_j(x_j) \equiv \sum_{j=1}^{n} \sum_{k=0}^{r_j} f_{jk}\lambda_{jk} \\
&\text{s.t.} \quad \sum_{j=1}^{n} \hat{g}_{ij}(x_j) \equiv \sum_{j=1}^{n} \sum_{k=0}^{r_j} g_{ijk}\lambda_{jk}\rho b_i \quad , i = 1,\dots,m, \\
&\qquad \sum_{k=0}^{r_j} \lambda_{jk} = 1 \qquad , j = 1,\dots,n \\
&\text{and} \quad \lambda_{jk} \geq 0 \ , \forall j \text{ and } k \qquad \rho \text{ is } \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix}
\end{aligned}\right\} (2.4)$$

and subject to the following restrictions for each j , $j = 1, \dots, n$

(i) At most two of the λ_{jk} can be positive , and

(ii) If two are positive , they must be adjacent .

The values of the variables x_j associated with any particular solution λ are given by

$$x_j = \sum_{k=0}^{r_j} \lambda_{jk} x_{jk} , \quad j = 1,\dots,n \qquad (2.5)$$

and hence it must be integer if x_{jk} are integers (theorem 1.1) . Observing that the approximating problem (2.4) is identical to the problem

$$\left.\begin{aligned}
&\text{Max } Z = \sum_{j=1}^{n} \hat{f}_j(x_j) \\
&\text{s.t.} \quad \sum_{j=1}^{n} \hat{g}_{ij}(x_j) \rho \ b_i \quad , i = 1,\dots,m, \\
&\text{and} \quad \underline{x}_j \leq x_j \leq \overline{x}_j \quad , j = 1,\dots,n
\end{aligned}\right\} (2.6)$$

which is itself an approximation to the original problem (2. 1), assuming that the lower and upper bounds $\underline{x}_j$ and $\overline{x}_j$ are in the feasible region.

3. Examples

To prove theorem 1.1. and applying the previous algorithm, we have to introduce the following two examples. The first example with x_{jk} all integers while the second one with x_{jk} are not all integers, for the same problem.

Example

3.1. Max $Z = (x_1-1)^2+(x_2-1)^2$

Subject to $x_1 + 2x_2 \leq 5$

$x_1, x_2 \geq 0$

Solve :-

$f_1(x_1) = (x_1-1)^2$, $f_2(x_2) = (x_2-1)^2$

Break Points of $f_1(x_1)$		Break Points of $f_2(x_2)$	
$x_{10}=0$	$f_{10}=1$	$x_{20}=0$	$f_{20}=1$
$x_{11}=1$	$f_{11}=0$	$x_{21}=1$	$f_{21}=0$
$x_{12}=2$	$f_{12}=1$	$x_{22}=2$	$f_{22}=1$
$x_{13}=3$	$f_{13}=4$		
$x_{14}=4$	$f_{14}=9$		
$x_{15}=5$	$f_{15}=16$		

Max $Z = \lambda_{10} + \lambda_{12} + 4\lambda_{13} + 9\lambda_{14} + 16\lambda_{15} + \lambda_{20} + \lambda_{22}$

Subject to

$\lambda_{11} + 2\lambda_{12} + 3\lambda_{13} + 4\lambda_{14} + 5\lambda_{15} + 2\lambda_{21} + 4\lambda_{22} \leq 5,$

$\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} = 1,$

$$\lambda_{20} + \lambda_{21} + \lambda_{22} \quad = \quad 1,$$
$$\lambda_{jk} \geq 0 \qquad \text{for all j and k}$$

Min $\acute{Z} = - \lambda_{10} - \lambda_{12} - 4\,\lambda_{13} - 9\,\lambda_{14} - 16\,\lambda_{15} - \lambda_{20} - \lambda_{22}$

The previous example becomes

$$\lambda_{11} + 2\,\lambda_{12} + 3\,\lambda_{13} + 4\,\lambda_{14} + 5\,\lambda_{15} + 2\,\lambda_{21} + 4\,\lambda_{22} + \lambda' = 5,$$
$$\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} + y_1 = 1,$$
$$\lambda_{20} + \lambda_{21} + \lambda_{22} + y_2 = 1,$$
$$- \lambda_{10} - \lambda_{12} - 4\,\lambda_{13} - 9\,\lambda_{14} - 16\,\lambda_{15} - \lambda_{20} - \lambda_{22} + (-\acute{Z}) = 0,$$
$$- \lambda_{10} - \lambda_{11} - \lambda_{12} - \lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{20} - \lambda_{21} - \lambda_{22} + (-w) = -2$$

Where λ' slack variable and y_1, y_2 artificial variables and

$$w = \sum_{j=1}^{2} y_i = y_1 + y_2$$

Phase 1

Basic	b	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{20}	λ_{21}	λ_{22}	λ'	y_1	y_2
-w	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
-Ź	0	-1	0	-1	-4	-9	-16	-1	0	-1	0	0	0
y_1	1	①	1	1	1	1	1	0	0	0	0	1	0
y_2	1	0	0	0	0	0	0	1	1	1	0	0	1
λ'	5	0	1	2	3	4	5	0	2	4	1	0	0
-w	0	0	0	0	0	0	0	0	0	0	0	1	1
-Ź	2	0	1	0	-3	-8	-15	0	1	0	0	1	1
λ_{10}	1	1	1	1	1	1	1	0	0	0	0	1	0
λ_{20}	1	0	0	0	0	0	0	1	1	1	0	0	1
λ'	5	0	1	2	3	4	5	0	2	4	1	0	0

since min w = 0

Then we reach to the optimal solution of w (i.e. the end of phase 1 and begin phase 2

Phase 2

Basic	b	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{20}	λ_{21}	λ_{22}	λ'
-Ź	2	0	1	0	-3	-8	-15	0	1	0	0
λ_{10}	1	1	1	1	1	1	①	0	0	0	0
λ_{20}	1	0	0	0	0	0	0	1	1	1	0
λ'	5	0	1	2	3	4	5	0	2	4	1
-Ź	17	15	16	15	12	7	0	0	1	0	0
λ_{15}	1	1	1	1	1	1	1	0	0	0	0
λ_{20}	1	0	0	0	0	0	0	1	1	1	0
λ'	0	-5	-4	-3	-2	-1	0	0	2	4	1

since all elements in the row -Ź either zero or positive then this is the optimal solution . (i.e.) ,

$\lambda_{10}=\lambda_{11}=\lambda_{12}=\lambda_{13}=\lambda_{14}=0$, $\lambda_{15}=1$,

$\lambda_{20}=1$, $\lambda_{21}=\lambda_{22}=0$, $\lambda'=0$, $Z'=-17$

$\lambda'\rightarrow z=17$

3.2. Max $Z=(x_1-1)^2+(x_2-1)^2$

Subject to $x_1+2x_2\le 5$

$x_1, x_2 \ge 0$

solve :-

$f_1(x_1)=(x_1-1)^2$, $f_2(x_2)=(x_2-1)^2$

Break Points of $f_1(x_1)$		Break Points of $f_2(x_2)$	
$x_{10}=0$	$f_{10}=1$	$x_{20}=0$	$f_{20}=1$
$x_{11}=\frac{1}{2}$	$f_{11}=\frac{1}{4}$	$x_{21}=\frac{1}{2}$	$f_{21}=\frac{1}{4}$
$x_{12}=1$	$f_{12}=0$	$x_{22}=\frac{3}{4}$	$f_{22}=\frac{1}{16}$
$x_{13}=1\frac{1}{2}$	$f_{13}=\frac{1}{4}$	$x_{23}=1$	$f_{23}=0$
$x_{14}=2$	$f_{14}=1$	$x_{24}=1\frac{1}{2}$	$f_{24}=\frac{1}{4}$
$x_{15}=3$	$f_{15}=4$	$x_{25}=1\frac{3}{4}$	$f_{25}=\frac{9}{16}$
$x_{16}=4$	$f_{16}=9$	$x_{26}=2$	$f_{26}=1$
$x_{17}=4\frac{1}{2}$	$f_{17}=\frac{49}{4}$		
$x_{18}=5$	$f_{18}=16$		

$$\max Z=\lambda_{10}+\frac{1}{4}\lambda_{11}+\frac{1}{4}\lambda_{13}+\lambda_{14}+4\lambda_{15}+9\lambda_{16}+\frac{49}{4}\lambda_{17}+16\lambda_{18}$$
$$+\lambda_{20}+\frac{1}{4}\lambda_{21}+\frac{1}{16}\lambda_{22}+\frac{1}{4}\lambda_{24}+\frac{9}{16}\lambda_{25}+\lambda_{26}$$

subject to

$$\frac{1}{2}\lambda_{11}+\lambda_{12}+\frac{3}{2}\lambda_{13}+2\lambda_{14}+3\lambda_{15}+4\lambda_{16}+\frac{9}{2}\lambda_{17}+5\lambda_{18}$$
$$+\lambda_{21}+\frac{3}{2}\lambda_{22}+2\lambda_{23}+3\lambda_{24}+\frac{7}{2}\lambda_{25}+4\lambda_{26}\le 5\ ,$$
$$\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18}=1\ ,$$
$$\lambda_{20}+\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{24}+\lambda_{25}+\lambda_{26}=1\ ,$$
$$\lambda_{jk}\ge 0 \qquad (\text{for all } j \text{ and } k)$$

$$\text{Min } Z'=-\lambda_{10}-\frac{1}{4}\lambda_{11}-\frac{1}{4}\lambda_{13}-\lambda_{14}-4\lambda_{15}-9\lambda_{16}-\frac{49}{4}\lambda_{17}-16\lambda_{18}$$
$$-\lambda_{20}-\frac{1}{4}\lambda_{21}-\frac{1}{16}\lambda_{22}-\frac{1}{4}\lambda_{24}-\frac{9}{16}\lambda_{25}-\lambda_{26}$$

The previous example becomes

$$\begin{aligned}
&\tfrac{1}{2}\lambda_{11} + \lambda_{12} + \tfrac{3}{2}\lambda_{13} + 2\lambda_{14} + 3\lambda_{15} + 4\lambda_{16} + \tfrac{9}{2}\lambda_{17} + 5\lambda_{18} \\
&+ \lambda_{21} + \tfrac{3}{2}\lambda_{22} + 2\lambda_{23} + 3\lambda_{24} + \tfrac{7}{2}\lambda_{25} + 4\lambda_{26} + \lambda' \qquad = 5, \\
&\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} \qquad + y_1 = 1, \\
&\lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} + \lambda_{26} \qquad + y_2 = 1, \\
&-\lambda_{10} - \tfrac{1}{4}\lambda_{11} - \tfrac{1}{4}\lambda_{13} - \lambda_{14} - 4\lambda_{15} - 9\lambda_{16} - \tfrac{49}{4}\lambda_{17} - 16\lambda_{18} \\
&\qquad - \lambda_{20} - \tfrac{1}{4}\lambda_{21} - \tfrac{1}{16}\lambda_{22} - \tfrac{1}{4}\lambda_{24} - \tfrac{9}{16}\lambda_{25} - \lambda_{26} \quad (-Z') = 0
\end{aligned}$$

$$\begin{aligned}
&- \lambda_{10} - \lambda_{11} - \lambda_{12} - \lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{19} - \lambda_{20} - \lambda_{21} - \lambda_{22} - \lambda_{23} \\
&- \lambda_{24} - \lambda_{25} - \lambda_{26} \qquad + (-w) = -2
\end{aligned}$$

Where λ' is a slack variable and y_1 , y_2 are artificial variables and

$$w = \sum_{i=1}^{2} y_i$$

Phase 1

Basic	b	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{20}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	λ_{25}	λ_{26}	λ'	y_1	y_2
-w	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
$-\acute{Z}$	0	-1	-1/4	0	-1/4	-1	-4	-9	-49/4	-16	-1	-1/4	-1/16	0	-1/4	-9/16	-1	0	0	0
y_1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0
y_2	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	1
λ'	5	0	1/2	1	3/2	2	3	4	9/2	5	0	1	3/2	2	3	7/2	4	1	0	0
-w	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$-\acute{Z}$	2	0	3/4	1	3/4	0	-3	-8	-45/4	-15	0	3/4	15/16	1	3/4	7/16	0	0	1	1
λ_{10}	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0
λ_{20}	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	1
λ'	5	0	1/2	1	3/2	2	3	4	9/2	5	0	1	3/2	2	3	7/2	4	1	0	0

Since min w = 0

Then we reach to the optimal solution of w(i.e. the end of phase 1 and begin phase 2)

Phase 2

Basic	b	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{20}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	λ_{25}	λ_{26}	λ'
-Ź	2	0	3/4	1	3/4	0	-3	-8	-45/4	-15	0	3/4	15/16	1	3/4	7/16	0	0
λ_{10}	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
λ_{20}	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0
λ	5	0	1/2	1	3/2	2	3	4	9/2	5	0	1	3/2	2	3	7/2	4	1
-Ź	17	15	63/4	16	63/4	15	12	7	15/4	0	0	3/4	15/16	1	3/4	7/16	0	0
λ_{18}	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
λ_{20}	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0
λ	0	-5	-9/2	-4	-7/2	-3	-2	-1	-1/2	0	0	1	3/2	2	3	7/2	4	1

Since all elements in the row – Z` either zero or positive, then this is the optimal solution (i.e.),

$$\lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} == \lambda_{15} = \lambda_{16} = \lambda_{17} = 0,\ \lambda_{18} = 1,$$

$$\lambda_{20} = 1,\ \lambda_{21} = \lambda_{22} = \lambda_{23} = \lambda_{24} = \lambda_{25} = \lambda_{26} = 0\ ,$$

$$\lambda'_{=0}\ ,\ Z' = -17 \rightarrow Z = 17.$$

REFERENCES

1- Abadia, J. ed. , Non-linear programming, North - Holland, Amsterdam, 1967.

2- Abadia, J. ed. , Integer and Non-linear Programming, North - Holland, Amsterdam,1970 .

3- Charnes, A. and C. lemke, "Minimization of Non-linear Separable Convex Functionals", Naval Res. Log. Quart. , I, no.4, Dec. 1954, pp. 301-312.

4- Fiacco,A.V.andG.P.McCormick," Computational Algorithm for the Sequential Unconstrained Minimization Technique for Non-linear Programming", Management Sci. , 10, no. 4 , July 1964, pp. 601-617 .

5- Fiacco,A.V.andG.P.McCorrmick, Non-linear Programming:SequentialUnconstrained Minimization Techniques , Wiley , New York , 1968 .

6- Garfinkel,R.andG.Nemhauser, Integer Programming, Wiley,NewYork, '1972.

7- Graves,R.and P.Wolfe,eds.,Recent Advances in Mathematical Programming, McGraw - Hill, New York, 1963.

8- Hadley, G., Non-linear and Dynamic Programming, Addison- wesley, Reading, Mass. , 1964.

9- Kelley,J.E.,"The Cutting-plane Method for Solving Convex programs", J.Soc. Ind. Appl. Math. , 8 , no. 4, Dec. 1960, PP. 703 - 712.

10- Kuhn, H. and A.W. Tucker, " Non-linear Programming", in Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability , J. Neyman, ed. , University of California Press , Berkeley , Calif . , 1951 , pp. 481 - 492.

11-Miller, C. E. , "The Simplex Method for local Separable programming", in Graves and Wolfe [7] , 1963 , PP. 89 - 1 00.

12-Salkin , H. , Integer Programming , Addison - Wesley , Reading , Mass. , 1975 .
13-Saaty , T. L. and J. Bram , Non-linear Mathematics , McGraw -Hill , New York , 1964.
14-Simmons,D.M.,Non-linear Programming, Prentice-Hall Inc., Englewood Cliffs, N. J. , 1975.
15-Taha , H. , Integer Programming Theory , Academic Press , New York , 1975 .
16-Tucker,A.W.,"Linear and Non-linear programming" ,operations Res.15,no.2, April 1975 , PP. 244 - 257.
17-Wolfe, P., "Methods of Non-linear Programming", in Abadie [1], 1967, PP. 97 131 .
18-Zangwill , W. I. , Non-linear Programming : A Unified Approach , Prentice - Hall , Englewood cliffs, N. J. , 1969.
19- Zoutendijk , G. , "Non-linear Programming : A Numerical Survey" , S. I. A. M. J. Control , 4, no. 1, Feb. 1966, pp. 194 - 210 .
20- Zoutendijk,G.," Nonlinear Prograinming, Computational Methods", in Abadie [2] , 1970 , PP. 3 7 - 86 .

الملخص العربي

البرمجة غير الخطية المنفصلة تم معالجتها في موضوعات البرمجة غير الخطية (تعريف البرمجة غير الخطية) في هذا البحث تم دراسة طريقة مقترحة لمسألة البرمجة غير الخطية الصحيحة، بدءا بمثالية لإثبات النظرية المقترحة في البحث.
حيث طرق البرمجة المنفصلة هي الأفضل لحل مسائل البرمجة المنفصلة المحدبة ولذا يمكن استخدامها لحل مسائل البرمجة غير المحدبة باستخدام الحاسب الآلي ذو السرعات العالية