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ON THE DIMENSION OF THE SPACES NEAR TO STRONG PARACOMPACTNESS

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ABSTRACT

In this paper we introduce the concept of TP-rarefied spaces and investigate some of its dimensional properties. We prove that $\dim Y \leq \dim X$ for any closed σ–totally paracompact subspace Y of a TP-rarefied space X. Moreover, we study some defects of σ–totally paracompact spaces.

INTRODUCTION

This aim of this paper is to study the relation between the dimension of the spaces and the dimension of their quassi components. This connection is given in the works of A. Lekek [5] and T. Nishura [7] for separable metric spaces. We extend these results over the limit of metrizable spaces.

BASIC DEFINITIONS AND NOTATIONS

The space X is strongly paracompact if every open cover of the space X has a star-finite open refinement. The space X is completely paracompact [4,9] if for every open cover U of the space X there exists a sequence $\psi_1, \psi_2, \ldots$ of star finite open covers of X such that the union $\bigcup_{i=1}^{\infty} \Psi_i$ contains a refinement of U. Thespace X is called σ–totally paracompact (briefly σ–t.p.) [6] if for every base $\Re$ for the space X there exists a σ–locally finite open cover ψ of X such that for each $V \in \psi$ one can find $U \in \Re$ such that $V \subset U$ and $Fr\ V \subset Fr\ U$. Thesymbol Fr U denotes the boundary of a set U in X. Every σ.t.p. space is paracompact and a completely paracompact space is σ.t. paracompact.

As usual by dim X and ind X we denote the covering dimension and the small inductive dimension respectively. N will denote the set of natural numbers. The local dimension loc dim X of a space X is defined as the least integer n such that there exists an open cover $\{U_\lambda\}$ of X with each dim $[U_\lambda] \leq n$, or if there is no such integer, loc dim = $= \infty$. The symbol ClU denotes the closure of a set U in X and when the closure of U is taken with respect to any space Y we denote it by $Cl_Y U$. Let $f:X \rightarrow Y$ be a mapping , then $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$.

The family $\{H_a : a \in A\}$ is called hereditarily conservative if $Cl (\cup \{L_a : a \in A\}) = \cup\{Cl\ L_a : a \in A\}$ for every family $\{L_a \subseteq H_a : a \in A\}$. The family ω is χ_0 – conseruative if $\omega = \cup \{\omega_n : n \in N\}$, where ω_0 is hereditarily conservative and closed in X, $(\cup\{L : L \in \omega_i\}) \cap (\cup \{ H : H \in \omega_j\}) = \varnothing$ for $i \neq j$ and for every $n \geq 1$ the family ω_n is hereditarily conservative and closed in $X \setminus \cup\{L : L \in \cup \{\omega_i : i \leq n-1\}\}$. The space X is called L–paracompact if every open cover of X has χ_0–conservative refinement. M.M. Coban [2,3] shows that Every L–paracompact space satisfies the condition:

(*) If Z is a closed subspace of the space X , then dim Z =
= loc dim Z.

All spaces are considered to be normal unless stated otherwise.

TP-RAREFIED SPACES

Definition 2.1. A space X is called TP–rarefied if for every nonempty closed subspace $Y \subset X$ there exists a nonempty open set U such that Cl U is a union of a countable number of closed σ–totally paracompact subspaces.

Theorem 2.2. If the space X is TP-rarefied and satisfies the condition (*) , then $\dim X = \sup\{\dim Y : Y$ is closed in X and Y is σ.t.p. space$\}$

Proof: Let $n = \sup\{\dim Y : Y$ is closed in X and Y is σ.t.p. space$\}$ and $Z = \cup\{U : U$ is open in X and $\dim U \leq n\}$. It is clear that $\dim X \geq n$. If $Y \subset Z$ and Y is closed in X, then by the condition (*) we have $\dim Y = \text{loc} \dim Y \leq \text{loc} \dim Z \leq n$. We shall prove that $Z = X$. Then $\dim X \leq n$ and the theorem is proved.Assume that $Z \neq$ $\neq X$. Then the set $X \setminus Z$ is closed and since X is TP–rarefied, then there exists a nonempty open in $X \setminus Z$ set U such that Cl U is σ–totally paracompact. The set $Z \cup U$ is open in X. Consider a point $x \in U$ and V is an open in X such that $x \in V \subset Y = \text{Cl}\, V \subset Z \cup U$. Then we have,

1. $Y \setminus Z \subset U$ and the set $Y \setminus Z$ is closed in X and σ–totally paracompact.

2. $\dim (Y \setminus Z) \leq n$,

3. If $H \subset Y \cap Z$ and H is closed in Y, then $H \subset Z$ and $\dim H = \operatorname{loc} \dim H \leq \operatorname{loc} \dim Z \leq n$.

By lemma 3.1.6 in [4] we have $\dim Y \leq \dim (Y \setminus Z) + \sup \{\dim H : H \text{ is closed in } Y \text{ and } H \subset Z\} \leq n$. Then $x \in V \subset Z$ and this contradicts the assumption that $x \notin Z$. From the contradiction we see that $Z = X$ and the theorem is proved.

Corollary 2.3. If the space X is TP-rarefied and L–paracompact, then $\dim X = \sup\{\dim Y : Y \text{ is closed in } X \text{ and } Y \text{ is } \sigma\text{–t.p. space}\}$

Theorem 2.4. If for a σ–totally paracompact space X there exists an open base $\mathfrak{R}$ such that $\dim \operatorname{Fr} U \leq n-1$ for all $U \in \mathfrak{R}$, then $\dim X \leq n$.

Proof: Let A and B be two disjoint closed sets in X. Let $\mathfrak{R}_1 = \{U \in \mathfrak{R} : A \cap \operatorname{Cl} U = \varnothing \text{ or } B \cap \operatorname{Cl} U = \varnothing\}$. Clearly $\mathfrak{R}_1$ is a base for the space X. Then there exist discrete open systems;

$$r_m = \left\{ \{U^m_a : a \in A_m\} : m \in N \right\} \text{ and } \left\{ V^m_\alpha \in \mathfrak{R}_1 : \alpha \in A_m,\ m \in N \right\}$$

in X such that

1. $r = \cup \{r_m : m \in N\}$ is open cover of the space X.
2. $\cup^m_\alpha \subset V^m_\alpha$ and $\operatorname{Fr} U^m_\alpha \subset \operatorname{Fr} V^m_\alpha$.

Let $A'_m = \left\{ \alpha \in A_m : A \cap \operatorname{Cl} U^m_\alpha \neq \varnothing \right\}$, $U_m = \cup$ $\cup \left\{ U^m_\alpha : \alpha \in A'_m \right\}$ and $V_m = \cup \left\{ U^m_\alpha : \alpha \in A_m \setminus A'_m \right\}$.

By the construction we have $\mathrm{Cl}\, U_m \cap \mathrm{Cl}\, V_m = \varnothing$, $\mathrm{Fr}\, U_m = \cup$ $\cup\{ \mathrm{F}\, U_\alpha^m : \alpha \in A'_m \}$ and $\mathrm{Fr}\, V_m = \cup \{\mathrm{Fr}\ U_\alpha^m : \alpha \in A_m \setminus A'_m \}$. Also $\dim \mathrm{Fr}\, U_m \leq n-1$ and $\dim \mathrm{Fr}\, V_m \leq n-1$. Let

$$G_m = U_m \setminus \cup \{\mathrm{Cl}\, V_i : i \leq m\}, \quad H_m = V_m \setminus \cup \{\mathrm{Cl}\, U_i : i \leq m),$$

$$G = \cup \{G_m : m \in N\}, \quad \text{and} \quad H = \cup \{ H_m : m \in N\}$$

Then $A \subset G$, $B \subset H$ and $G \cap H = \varnothing$. Also $X \setminus (G \cup H) \subset \cup$ $\cup\{\mathrm{Fr}\, G_m \cup \mathrm{Fr}\, H_m : m \in N\}$. Then $\dim (X \setminus (G \cup H)) \leq n-1$. By Lemma 3.1.27 in [4] we have $\dim X \leq n$.

Corollary 2.4. If the space X is σ–totally paracompact, then $\dim X \leq \mathrm{ind}\, X$.

Corollary 2.5. If the space X is TP–rarefied and satpsfies the condition (*), then $\dim X \leq \mathrm{ind}\, X$.

Corollary 2.6. If the space X is L–paracompact and TP–rarefied, then $\dim X \leq \mathrm{ind}\, X$.

INDUCTIVE COMPACTNESS AND DEFECTS OF SPACES

Inductive dimensions are introduced by a class of spaces and investigated in [1,8].

Let $\mathcal{B}$– be a class of spaces. For any space X we define,

(1) $\mathcal{B}$ - Ind X = −1 and $\mathcal{B}$ - ind X = −1 if $X \in \mathcal{B}$.

(2) If $n \geq 0$, then $\mathcal{B}$–Ind $X \leq n$, if for every closed set F in X and an open set U in X such that $U \supset F$ there exists an open set V in X such that $F \subset V \subset U$ and $\mathcal{B}$– Ind $\mathrm{Fr}V \leq n-1$.

(3) If $n \geq 0$, then $\mathcal{B}$–ind $X \leq n$, if for every point x in X and every neighbourhood O_x of the point x there exists a neighbourhood V_x of x such that $V_x \subset O_x$ and $\mathcal{B}$–ind Fr $V_x \leq n-1$.

If K is the class of all compact spaces, then K-ind X = cmp X and K-Ind X = Cmp X are called the inductive compactness of the space X.

Let $\dim_{\mathcal{B}} X = \sup\{\dim F : F$ is closed subset of X and $F \in \mathcal{B}\}$, and $\dim_K X = \sup\{\dim F$: F is compact subset of X$\}$.

By Lemma 3.1.27 in [4] we have .

Corollary 3.1. For every space X we have $\dim X \leq \mathcal{B}$–Ind $X + \dim_{\mathcal{B}} X + 1$, and $\dim X \leq$ Cmp $X + \dim_K X + 1$.

By proposition 2.4 we have.

Corollary 3.2. If the space X satisfies one of the following:

1. The space X is σ –totally paracompact.

2. The space X is TP–rarefied and L–paracompact.

3. The space X is TP–rarefied and satisfies the condition (*)

, then $\dim X \leq \mathcal{B}$–ind $X + \dim_{\mathcal{B}} X + 1$ and $\dim X \leq$ Cmp $X + \dim_K X + 1$

Example 3.3. Let X be a metric space such that $\dim X = n \geq 2$ and $\dim F = 0$ for every compact subset F of the space X. In the space $X \times I$, where $I = [0,1]$ combine one point a with a set

$X \times \{1\}$. The resulting space is denoted by Z. Let $f : X \times I \to Z$ be a natural projection. If $z \in Z$ and $z \neq a$, then the neighbourhood O_z of the point z in Z is as in the space $X \times I$. If $z = a$, then the neighbourhood of a has the form $O_a = f(X \times (1-\varepsilon, 1])$, $\varepsilon > 0$. The space X is metrizable, linearly connected, $n \leq \dim Z \leq n+1$ and $\dim_K Z=1$. Thus $\dim Z - \dim_K Z$ can be arbitrarily a large number even in the class of separable metric spaces.

By Q(x,X) we denote a quasi component of a point x in the space X. There exists a continuous mapping $q_X : X \to X \mid Q$ where $q_X(x)= Q(x, X)$. For every point $x \in X$ on $X \mid Q$ we consider the topology generateed by a base $\{U \subset X/Q :$ the set $q_X^{-1}(U)$ is open–closed in $X\}$. The nspace $X \mid Q$ is called quasi component space and q_X is the natural projection onto $X \mid Q$.

By Co(X) we denote the family of all compactifications of the space X. Also C(x,X) is the connected component of a point x in the space X. If $Y \subset X$, then we define

$$r\, d_X Y = \sup\{\dim F : F \subset Y \text{ and } F \text{ is closed in } X\}.$$

Also for the space X we define the following defects,

$$\operatorname{def} X = \inf\{\dim(CX \setminus X) : CX \in Co(X)\}$$

and $$\operatorname{Def} X = \inf\{rd_{CX}(CX \setminus X) : CX \in Co(X)\}.$$

The quasi dimension and quasi locally dimension of the space X are defined as follows;

On the dimension

$$Q \dim X = \sup \{\dim Q(x,X) : x \in X\}$$

and

$$Q \operatorname{loc} \dim X = \sup \{\operatorname{loc} \dim Q(x,X) : x \in X\}$$

Lemma 3.4. If the space X is locally compact, then $\operatorname{loc} \dim X \leq$ $\leq \dim_C X = \sup\{\dim C(x,X) : x \in X\}$.

Proof: Let x be a point of the space X and U be a neighbourhood of the point x such that $F = \mathrm{Cl}\, U$ is compact. Then $\dim F = Q \dim F = \dim_C F \leq \dim_C X$. Then $\operatorname{loc} \dim X \leq \dim_C X$.

Consider the following condition:

(**) For the space X there exists $bX \in Co(X)$ such tthat $Q(x,X) = X \cap Q(x,bX)$ for every $x \in X$.

Lemma 3.5. If the space X satisfies the condition (**) and $Q(x, X)$ is locally compact for every $x \in X$, then $\dim bX =$ $= rd_{bX} (bX \setminus X) + Q \operatorname{loc} \dim X$.

Proof: Let $y \in bX$ be any point, where bX is the compactification of the sapce X stated in the condition (**). If $Q(y,bX) \subset bX \setminus X$, then $\dim Q(y,bX) \leq \leq rd_{bX} (bX \setminus X)$. Let us suppose that $X \cap Q(y, bX) \neq \varnothing$ and $y \in X$. The set $Q(y,X)$ is locally compact and open in $Q(y, bX)$. Then $\dim Q(y, bX) =$ $= \dim (Q(y, bX) \setminus Q(y, X)) + \operatorname{loc} \dim Q(y,X) \leq rd_{bX} (bX \setminus X) + Q$ $\operatorname{loc} \dim X$. For the mapping $q_{bX} : bX \to bX \mid Q$ we have $\dim$ $q_{bX} = Q \dim bX$ and $\dim bX / Q = 0$. Thus $\dim q_{bX} = Q \dim bX =$ $= rd_{bX} (bX \setminus X) + Q \operatorname{loc} \dim X$. and the proof is complete.

In the following two theorems all considered spaces are compact.

Theorem 3.6. If any compactification of a σ.t.p. space X is TP–rarefied and L–paracompact, then $\dim X \leq \operatorname{def} X + Q \dim X + 1$

Proof: Let bX be a compactification of the space X such that $\operatorname{def} X = \dim (bX \setminus X)$. Let $x \in X$ be any point. Then $\dim Q(x, bX) \leq \dim Q(x,X) + + \dim Q(x,bX \setminus X) + 1 \leq \operatorname{def} X + Q \dim X+1$. Thus $Q \dim bX \leq \operatorname{def} X + Q \dim X +1$. Hence $\dim bX \leq \leq \operatorname{def} X + Q \dim X + 1$. By Corollary 2.3. we have $\dim X \leq \operatorname{def} X + + Q \dim X + 1$

Theorem 3.7. Suppose that any compactification of a σ–t.p. space is TP–rarefied and L-paracompact. Also if the space X satisfies the condition (**) and Q(x, X) is locally compact for every point $x \in X$, then $\dim X \leq Q \dim X + \operatorname{Def} X$.

Proof: By Lemma 3.6 there exists $bX \in Co(X)$ such that

$\dim bX = rd_{bX}(bX \setminus X) + Q \dim X$ and $\operatorname{Def} X = rd_{bX}(bX \setminus X)$.

Then $\dim bX = \operatorname{Def} X + Q \dim X$. By Corollary 2.3. we have $\dim X \leq \leq \operatorname{Def} X + Q \dim X$.

Remark 3.8. Theorems 3.6. and 3.7 are given in [5,7] for separable metric spaces.

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أبعاد الفراغات القريبة من ثنائية الإحكام القوية

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يتناول هذا البحث دراسة مفهوم الفراغات المخلخلة من النوع -TP ودراسة بعض خواصها البعدية. كما تم دراسة أنواع الحيود المختلفة للفراغات التى لها محكمات من النوع مخلخل - TP