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STEADY STATE OF AVAILABILITY OF MULTIPLEX SYSTEMS

Abdel-Mohsen M. Metwally, M.O. Shaker**, A.E.A. Dessofy**, and M.A. El-Damcese***

* Nuclear Engineering Department, Faculty of Engineering, Alexandria University

**Mathematics Department, Faculty of Science, Tanta University.

ABSTRACT

In most practical applications, the constituent componenets or a multiplex system are usually unidentical. The present work study the solutions of the limiting values of transition probabilities to get the long-term availability for a multiplex system of n repairable identical/ unidentical components. Availability formulae for some selected configurations of definite number of components are achieved,

1-INTRODUCTION

In Markovian processes where it is possible to go from one state to another over a large long period of time, it can easily be shown that the limit $P_j = \lim_{t \to \infty} P_j(t)$ always exists[1],, where $p_j(t)$ is the occurrence probability of j failures at time t.

One can get the steady state (long-term) solutions by simply setting all the derivatives $\dot{P}_j(t)$ equal zero, and hence the system of differential equations will be reduced to an equivalent system of algebraic equations. To solve these equations of n- components multiplex system, one can make use of :

$$\sum_{j=0}^{2^n-1} p_j = 1 \quad \text{...(1)}$$

All previous works in this respect deal with identical components which is not the case in most practical applications. The present work gives the transition matrix and solves all resulting algebraic equations to get the long-term availability, with repair, for n-unidentical components. The case of identical components is obtained as a special case.

2 -MARKOV STATE DEFINITION

To fully describle all possible system-states over a large long period of time, one should define the following terms [2] :

* Total number of states = $\sum_{i=0}^{2^n-1} S_i$

* Operating components $=(x_1, x_2, x_3 ...., x_{n-1}, x_n)$

* Non operating components $=(\bar{x}_1, \bar{x}_2, \bar{x}_3 ...., \bar{x}_{n-1}, \bar{x}_n)$

The following are some samples of the states under the previous notations :

$S_0 = x_1\ x_2\ x_3 \cdots x_{n-1}\ x_n \quad , \quad S_1 = \bar{x}_1\ x_2\ x_3 \cdots x_{n-1}\ x_n$

$S_2 = x_1\ \bar{x}_2\ x_3 \cdots x_{n-1}\ x_n \quad , \quad S_3 = x_1\ x_2\ \bar{x}_3 \cdots x_{n-1}\ x_n$

$\vdots$

$S_{2^n-2} = x_1\ \bar{x}_2\ \bar{x}_3 \cdots \bar{x}_{n-1}\ \bar{x}_n \quad , \quad S_{2^n-1} = \bar{x}_1\ \bar{x}_2\ \bar{x}_3 \cdots \bar{x}_{n-1}\ \bar{x}_n$

The state S_{2^n-2} for example, is the state where x_1 is only operating while the rest of components are down with. only one element can be under repair at a time.

The set of all system-states (Ω) is partitioned into the set of operational system-states (U) and the set of fialed system-states(D). One must write $S \varepsilon U$ when the system is operational [3] ·

System availability over a large long period of time is $A(\infty)=Pr\{S \varepsilon U$ over a large long period of time$|S \varepsilon \Omega$ at $t=0\}$. .... (2)

3 - BASIC ASSUMPTIONS

The solutions will be carried out under the following assumptions [4] :

i - The transition probability $P_{i,j}$ from S_i to S_j over a large long period of time, which includes exactly one failing element, is λ_i . λ_i is the failure rate of the ith components $i=1,2,\ldots\ldots\ldots n$.

ii - The transition probability $P_{i,j}$ from S_i to S_j over a large long period of time, which includes exactly one repair completion, is μ_i . μ_i is the repair rate of the ith components, $i=1,2,\ldots\ldots\ldots,n$.

iii -The transition probability $P_{i,j}$ from S_i to S_j over a large long period of time, in which the number of failure and repair completions exceeds one, is zero.

4 - MATHEMATICAL MODEL

A. Multiplex System of n-unidentical Components

The transition probablility matrix for availability with repair of n-unidentical components is given by :

$$P = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3\ \cdots & 2^n-2 & 2^n-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3\ \cdots \\ 2^n-2 \\ 2^n-1 \end{array} & \begin{bmatrix} 1-\sum_{i=1}^{n}\lambda_i & \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 \\ \mu_1 & 1-\sum_{i=2}^{n}\lambda_i-\mu_1 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 1-\sum_{i=1}^{n}\lambda_i+\lambda_2-\mu_2 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 1-\sum_{i=1}^{n}\lambda_i+\lambda_3-\mu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-(\lambda_1+\sum_{i=2}^{n}\mu_i) & \lambda_1 \\ 0 & 0 & 0 & 0 & \mu_1 & 1-\sum_{i=1}^{n}\mu_i \end{bmatrix} \end{array}$$

We may write the following system of 2^n algebraic equations are :

$$0 = -(\sum_{i=1}^{n}\lambda_i)P_o + \sum_{i=1}^{n}\mu_i P_i \qquad \text{(one equation)} \qquad (3.a)$$

$$0 = \lambda_j P_o - [\sum_{i=1}^{n}\lambda_i - \lambda_j + \mu_j]P_j + \sum_{\ell=1}^{n-1}\mu_{j,\ell} P_{j,\bar{\ell}}\ , \qquad \text{(n-equations)}(3.b)$$

$$j = 1,2,\ldots,n$$

$$0 = \sum_{k=1}^{m}\lambda_{j,\bar{k}} P_{j,k} - [\sum_{\ell=1}^{n-m}\lambda_{j,\ell} + \sum_{k=1}^{m}\mu_{j,\bar{k}}]P_j + \sum_{\ell=1}^{n-m}\mu_{j,\ell} P_{j,\bar{\ell}}\ .$$

$$\cdot\ (2^n-(n+1)\ \text{equations})(3.c)$$

$$m = 2,3,4,\ldots,n$$

$$j = [\sum_{i=1}^{m-1}\binom{n}{i}] + 1,\ldots,\sum_{i=1}^{m}\binom{n}{i}$$

where

$\lambda_{j,\bar{k}}$, $\mu_{j,\bar{k}}$ ≡ failure rate, repair rate of $\bar{k}$th bad component in state S_j

$\lambda_{j,\ell}$, $\mu_{j,\ell}$ ≡ failure rate, repair rate of ℓth good component in state S_j

$P_{j,k}$ ≡ state probability S_j when the kth bad component is replaced by a good one

$P_{j,\bar{\ell}}$ = state probability S_j when the ℓ^{th} good component is replaced by a bad one

n = total number of components

m = number of bad components in the state S_j

and the solutions are :

$$P_j = [\prod_{k=1}^{m} \frac{\lambda_{j,\bar{k}}}{\mu_{j,\bar{k}}}]P_o \quad , \quad \begin{array}{l} j = 1,2,\ldots,2^n-1 \\ m = 1,2,\ldots,n \end{array} \quad (4.a)$$

where

$$P_o = 1/[\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i} + \prod_{i=1}^{n} \frac{\lambda_i}{\mu_i} + \sum_{m=2}^{n-1} \sum_{\ell=\binom{n}{m-1}+1}^{\binom{n}{m}} (\prod_{k=1}^{m} \frac{\lambda_{j,\bar{k}}}{\mu_{j,\bar{k}}}) + 1], \quad (4.b)$$

$\binom{n}{m}$ is the combinational formula $\frac{n!}{(n-m)!\,m!}$

B - Multiplex System of n-identical Components

The set of algebraic equations can be obtained by replacing λ_i , μ_i by λ, μ respectively in the transition probbility matrix to get :

$$0 = -n\lambda P_o + n\mu P_1 \; , \quad \text{(one equation)} \quad (5.a)$$

$$0 = \lambda P_o - [(n-1)\lambda + \mu]P_j + (n-1)\mu P_{n+1} , \quad \text{(n-equations)} \quad (5.b)$$

$$j = 1,2,3,\ldots,n$$

$$0 = m\lambda P_{j-1} - [(n-m)\lambda + m\mu]P_j + (n-m)\mu P_{\ell} \; . \; (2^n-(n+1) \text{ equations}) (5.c)$$

$$m = 2,3,\ldots,n$$

$$j = [\sum_{i=1}^{m-1} \binom{n}{i}] + 1$$

$$\ell = [\sum_{i=1}^{m} \binom{n}{i}] + 1$$

where

$$P_j = P_{j+1} = \dots = P_k \; ; \qquad k = \sum_{i=1}^{m} \binom{n}{i}$$

In this case, the solutions are :

$$P_j = \left(\frac{\lambda}{\mu}\right) P_o \;, \qquad j = 1,2,3,\dots,n \tag{6.a}$$

and

$$P_j = \left(\frac{\lambda}{\mu}\right)^m P_o , \quad m=2,3,\dots,n; \; j = \left[\sum_{i=1}^{m-1} \binom{n}{i}\right] + 1, \dots, \sum_{i=1}^{m} \binom{n}{i} \tag{6.b}$$

where

$$P_o = \frac{1}{\sum_{i=o}^{n} \binom{n}{i} \left(\frac{\lambda}{\mu}\right)^i} \; . \tag{6.c}$$

C. Availability Formulae For Some Configurations:

C. I. Multiplex system of four unidentical components

One must define 2^4 states s_i, $i = 0,1,2,\dots\dots\dots\dots, 15$ over a large long period of time to fully describe all the component configurations associated with 4 element systems, the 16 states are :

$$
\begin{array}{llll}
s_o = x_1 x_2 x_3 x_4 , & s_1 = \bar{x}_1 x_2 x_3 x_4 , & s_2 = x_1 \bar{x}_2 x_3 x_4 , & s_3 = x_1 x_2 \bar{x}_3 x_4 \\
s_4 = x_1 x_2 x_3 \bar{x}_4 , & s_5 = \bar{x}_1 \bar{x}_2 x_3 x_4 , & s_6 = \bar{x}_1 x_2 \bar{x}_3 x_4 , & s_7 = \bar{x}_1 x_2 x_3 \bar{x}_4 \\
s_8 = x_1 \bar{x}_2 \bar{x}_3 x_4 , & s_9 = x_1 \bar{x}_2 x_3 \bar{x}_4 , & s_{10} = x_1 x_2 \bar{x}_3 \bar{x}_4 , & s_{11} = \bar{x}_1 \bar{x}_2 \bar{x}_3 x_4 \\
s_{12} = \bar{x}_1 \bar{x}_2 x_3 \bar{x}_4 , & s_{13} = \bar{x}_1 x_2 \bar{x}_3 \bar{x}_4 , & s_{14} = x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 , & s_{15} = \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4
\end{array}
$$

The system of 2^4 algebraic equations can be found from equations (3,a)-(3,c) where n=4 as :

$$o = -\left(\sum_{i=1}^{4} \lambda_i\right)p_o + \sum_{i=1}^{4} \mu_i\ p_i \tag{7.1}$$

$$o = \lambda_1\ p_o - \left(\sum_{i=2}^{4} \lambda_i + \mu_1\right)p_1 + \mu_2\ p_5 + \mu_3\ p_6 + \mu_4\ p_7 \tag{7.2}$$

$$o = \lambda_2\ p_o - \left(\sum_{i=1}^{4} \lambda_i - \lambda_2 + \mu_2\right)p_2 + \mu_1\ p_5 + \mu_3\ p_8 + \mu_4\ p_9 \tag{7.3}$$

$$o = \lambda_3\ p_o - \left(\sum_{i=1}^{4} \lambda_i - \lambda_3 + \mu_3\right)p_3 + \mu_1\ p_6 + \mu_2\ p_8 + \mu_4\ p_{10} \tag{7.4}$$

$$o = \lambda_4\ p_o - \left(\sum_{i=1}^{3} \lambda_i - \mu_4\right)p_4 + \mu_1 p_7 + \mu_2\ p_9 + \mu_3\ p_{10} \tag{7.5}$$

$$o = \lambda_1\ p_2 + \lambda_2\ p_1 - (\lambda_3 + \lambda_4 + \mu_1 + \mu_2)p_5 + \mu_3\ p_{11} + \mu_4\ p_{12} \tag{7.6}$$

$$o = \lambda_1\ p_3 + \lambda_3\ p_1 - (\lambda_2 + \lambda_4 + \mu_1 + \mu_3)p_6 + \mu_2\ p_{11} + \mu_4\ p_{13} \tag{7.7}$$

$$o = \lambda_1\ p_4 + \lambda_4\ p_1 - (\lambda_2 + \lambda_3 + \mu_1 + \mu_4)p_7 + \mu_2\ p_{12} + \mu_3\ p_{13} \tag{7.8}$$

$$o = \lambda_2\ p_3 + \lambda_3\ p_2 - (\lambda_1 + \lambda_4 + \mu_2 + \mu_3)p_8 + \mu_1\ p_{11} + \mu_4\ p_{14} \tag{7.9}$$

$$o = \lambda_2\ p_4 + \lambda_4\ p_2 - (\lambda_1 + \lambda_3 + \mu_2 + \mu_4)p_9 + \mu_1\ p_{12} + \mu_3\ p_{14} \tag{7.10}$$

$$o = \lambda_3\ p_4 + \lambda_4\ p_3 - (\lambda_1 + \lambda_2 + \mu_3 + \mu_4)p_{10} + \mu_1\ p_{13} + \mu_2\ p_{14} \tag{7.11}$$

$$o = \lambda_1\ p_8 + \lambda_2\ p_6 + \lambda_3\ p_5 - (\lambda_4 + \mu_1 + \mu_2 + \mu_3)p_{11} + \mu_4\ p_{15} \tag{7.12}$$

$$o = \lambda_1\ p_9 + \lambda_2\ p_7 + \lambda_4\ p_5 - (\lambda_3 + \mu_1 + \mu_2 + \mu_4)p_{12} + \mu_3\ p_{15} \tag{7.13}$$

$$o = \lambda_1\ p_{10} + \lambda_3\ p_7 + \lambda_4\ p_6 - (\lambda_2 + \mu_1 + \mu_3 + \mu_4)p_{13} + \mu_2\ p_{15} \tag{7.14}$$

$$o = \lambda_2\ p_{10} + \lambda_3\ p_9 + \lambda_4\ p_8 - (\lambda_1 + \mu_2 + \mu_3 + \mu_4)p_{14} + \mu_1\ p_{15} \tag{7.15}$$

$$o = \lambda_1\ p_{14} + \lambda_2\ p_{13} + \lambda_3\ p_{12} + \lambda_4\ p_{11} - \left(\sum_{i=1}^{4} \mu_i\right)p_{15} \tag{7.16}$$

and, from (1) we get :

$$1 = \sum_{j=o}^{15} p_j \tag{7.17}$$

The solution of this system of equations gives the following state probabilities :

$$P_o = 1/[\sum_{i=1}^{4} \frac{\lambda_i}{\mu_i} + \prod_{i=1}^{4} \frac{\lambda_i}{\mu_i} + (\frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} + \frac{\lambda_1 \lambda_3}{\mu_1 \mu_3} + \frac{\lambda_1 \lambda_4}{\mu_1 \mu_4} + \frac{\lambda_2 \lambda_3}{\mu_2 \mu_3} + \frac{\lambda_2 \lambda_4}{\mu_2 \mu_4} + \frac{\lambda_3 \lambda_4}{\mu_3 \mu_4} + \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3} + \frac{\lambda_1 \lambda_2 \lambda_4}{\mu_1 \mu_2 \mu_4} + \frac{\lambda_1 \lambda_3 \lambda_4}{\mu_1 \mu_3 \mu_4} + \frac{\lambda_2 \lambda_3 \lambda_4}{\mu_2 \mu_3 \mu_4}) + 1] \quad (8.a)$$

$$P_j = \frac{\lambda_j}{\mu_j} P_o \quad , \ j = 1,2,3,4 \quad (8.b)$$

$$P_j = \frac{\lambda_1 \lambda_k}{\mu_1 \mu_k} P_o \quad , \ j = 5,6,7 \ ; \ k = 2,3,4 \ \text{respectively} \quad (8.c)$$

$$P_j = \frac{\lambda_2 \lambda_k}{\mu_2 \mu_k} P_o \quad , \ j = 8,9 \ ; \ k = 3,4 \ \text{respectively} \quad (8.d)$$

$$P_{10} = \frac{\lambda_3 \lambda_4}{\mu_3 \mu_4} P_o \quad (8.e)$$

$$P_j = \frac{\lambda_1 \lambda_k \lambda_{k+1}}{\mu_1 \mu_k \mu_{k+1}} P_o \ , \quad j=11,12 \ ; \ k = 2,3 \ \text{respectively} \quad (8.f)$$

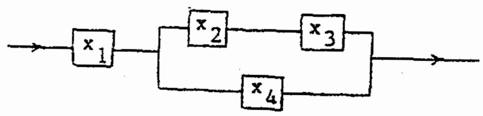
$$P_{13} = \frac{\lambda_1 \lambda_2 \lambda_4}{\mu_1 \mu_2 \mu_4} P_o \ , \quad P_{14} = \frac{\lambda_2 \lambda_3 \lambda_4}{\mu_2 \mu_3 \mu_4} P_o \ , \quad P_{15} = [\prod_{i=1}^{4} \frac{\lambda_i}{\mu_i}] P_o \quad (8.g)$$

The availability formulae depends on the configuration of the system. The following are some selected configurations :

(a)

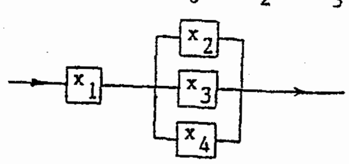
$$A(\infty) = P_o$$

(b)



$$A(\infty) = P_o + P_2 + P_3 + P_4 + P_8$$

(c)



$$A(\infty) = P_o + P_2 + P_3 + P_4 + P_8 + P_9 + P_{10}$$

C.2. Multiplex systm 10-identical components

The system of 2^{10} algebraic equations can be found from equations (6.a)-(6.b) when n=10 as :

$$o = -10\lambda p_o + 10\mu p_1 \qquad (9.1)$$

$$o = \lambda p_o - [9\lambda + \mu]p_j + 9\mu p_{11}, \qquad j=1,2,\ldots,10 \qquad (9.2)$$

$$o = 2\lambda p_{10} - [8\lambda + 2\mu]p_j + 8\mu p_{56}, \qquad j=11,12,\ldots,55 \qquad (9.3)$$

$$o = 3\lambda p_{55} - [7\lambda + 3\mu]p_j + 7\mu p_{176}, \qquad j=56,57,\ldots,175 \qquad (9.4)$$

$$o = 4\lambda p_{175} - [6\lambda + 4\mu]p_j + 6\mu p_{386}, \qquad j=176,177,\ldots,385 \qquad (9.5)$$

$$o = 5\lambda p_{385} - [5\lambda + 5\mu]p_j + 5\mu p_{638}, \qquad j=386,387,\ldots,637 \qquad (9.6)$$

$$o = 6\lambda p_{637} - [4\lambda + 6\mu]p_j + 4\mu p_{848}, \qquad j=638,639,\ldots,847 \qquad (9.7)$$

$$o = 7\lambda p_{847} - [3\lambda + 7\mu]p_j + 3\mu p_{968}, \qquad j=848,849,\ldots,967 \qquad (9.8)$$

$$o = 8\lambda p_{967} - [2\lambda + 8\mu]p_j + 2\mu p_{1013}, \qquad j=968,969,\ldots,1012 \qquad (9.9)$$

$$o = 9\lambda p_{1012} - [\lambda + 9\mu]p_j + \mu p_{1023}, \qquad j=1013,1014,\ldots,1022 \qquad (9.10)$$

$$o = 10\lambda p_{1022} - 10\mu p_{1023} \qquad (9.11)$$

and, from (1) we get

$$1 = \sum_{j=o}^{1023} p_j . \qquad (9.12)$$

The solution of this system of equations gives the following state probablities are :

$$p_o = \frac{1}{\sum_{i=o}^{10} \binom{10}{i}(\lambda/\mu)^i} \qquad (10.a)$$

$$p_j = (\lambda/\mu)p_o , \qquad j=1,2,\ldots,10 \qquad (10.b)$$

$$p_j = (\lambda/\mu)^m \, p_o , \qquad m=2,3,\ldots 10 \qquad (10.c)$$

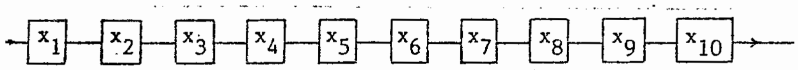
$$j=[\sum_{i=1}^{m-1} \binom{10}{i}]+1$$

where

$$p_j = p_{j+1} = \cdots = p_k , \qquad k=\sum_{i=1}^{m} \binom{10}{i} \qquad (10.d)$$

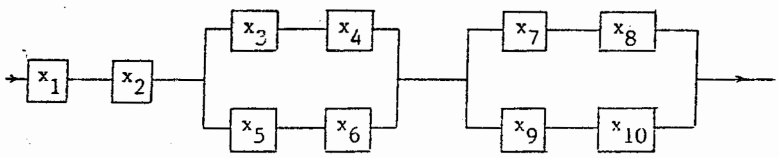
Shown below the availability formulae for some selecated configurations

(a)



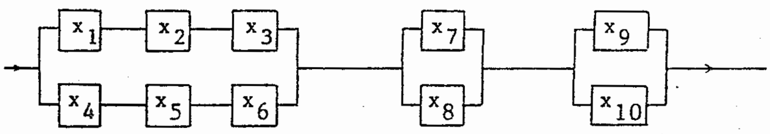
$$A(\infty) = P_o$$

(b)



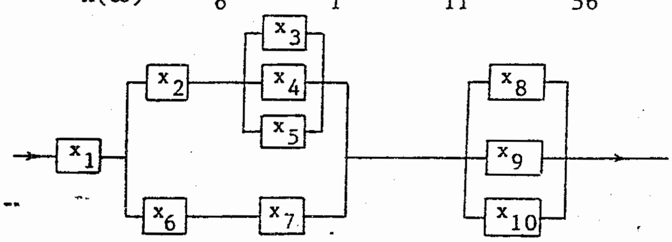
$$A(\infty) = P_o + 8\,P_1 + 20\,P_{11} + 16\,P_{56} + 4\,P_{176}$$

(c)



$$A(\infty) = P_o + 10\,P_1 + 34\,P_{11} + 50\,P_{56} + 32\,P_{176}$$

(d)



$$A(\infty) = P_o + 9\,P_1 + 34\,P_{11} + 70\,P_{56} + 82\,P_{176} + 51\,P_{386} + 12\,P_{638}$$

5 -The Utilization Factor (μ/λ)

The steady state availability is strongly affected by the ratio (μ/λ) which is commonly termed as " utilization factor" , some references refer the inverse of that ratio as "Operability ratio" .

Figure (1) shows how $A(\infty)$ varies with the utilization factor of the critical component x_1 (μ_1/λ_1) keeping other factors constant, for the three configurations of 4-unidentical component system presented in section (C.1.).

The same effect was studied for the configurations of 10-identical component system provided in section (C.2). The improvement in $A(\infty)$ as (μ/λ) increases is shown in Figure(2).

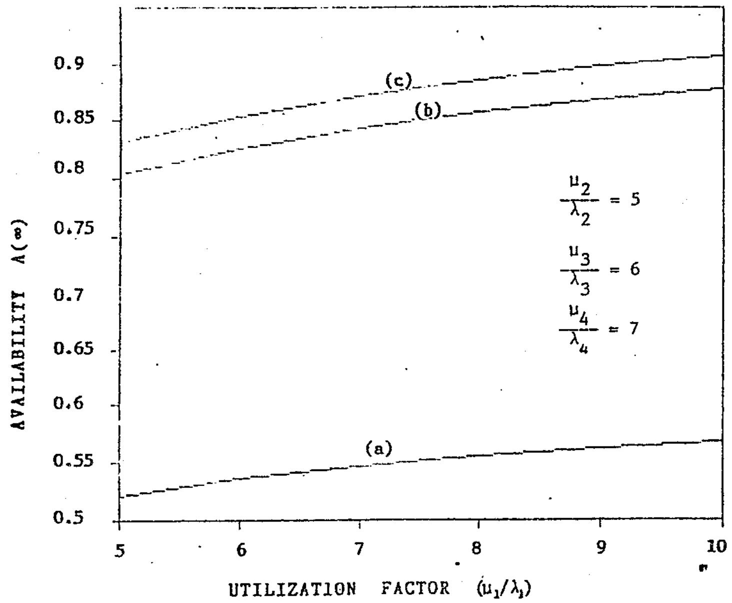


Figure (1) : Effect of critical component utilization factor on $A(\infty)$ response for different configurations.

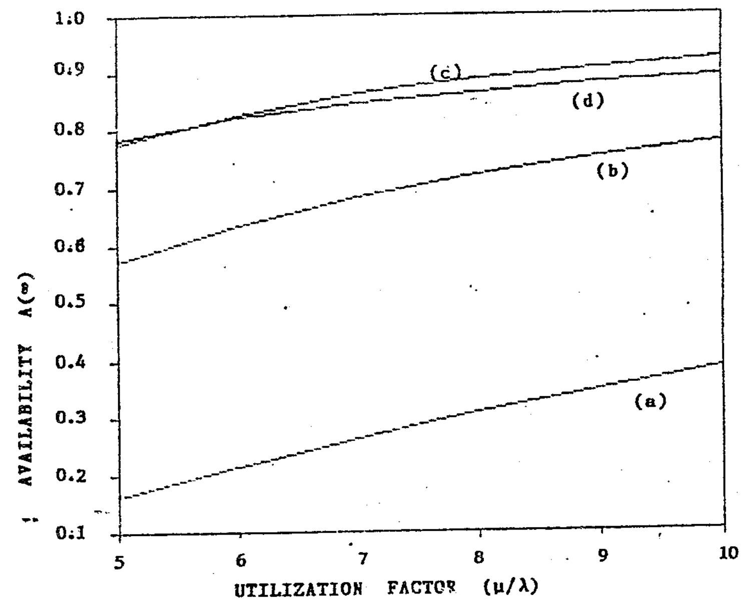


Figure (2) : Improvement in $A(\infty)$ of 10-identecal components response for different configurations.

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المتاحية النهائية للأنظمة المضاعفة

أ.د. عبد المحسن متولى ، أ.د. محمد عمر شاكر
د. عبد الله دسوقى ، مدحت الدمسيسى

فى معظم التطبيقات العملية تكون مكونات الأنظمة المضاعفة غير متماثلة . ويقدم هذا البحث دراسة لحلول القيم النهائية لاحتمالات الانتقال من حالة لأخرى . تم حساب المتاحية النهائية لنظام مضاعف يحتوى على ن مكون غير متماثل . أيضا تم اختيار بعض التشكيلات الهندسية واستنباط قوانين المتاحية النهائية لها .