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GROUPS OF HOMOLIOGY AND HOMOTOPY OF

Θ- TOPOLOGY

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INTRODUCTION

Our aim in this paper is to describe the groups of homology and the fundamental group π_1 of the Θ-topology, RΘR, which is the finest topology on the ground R^2 that induces the Euclidean topology on every line, ℓ_θ; with slope $\theta \in \Theta \subseteq \{\theta: 0 \leq \theta < \pi\}$, this will take place insection 3. In section 2 we study some topological properties of the Θ-topology and show some differences between this topology and the Euclidean topology RπR

2. Connectness and Countability

It is obvious that R π R satisfies the definition of RΘR whatever Θ is (even $\Theta = \{\theta : 0 < \theta < \pi\}$) i.e. each open set in Euclidean topology is open in the Θ-topology, hence the later one is fimer than the other. So the RΘR is Hausdorff.

2.1 Theorm

R θ R is pathwise connected if and only if θ has more than one element.

Proof

Let $x \neq y$ in R^2 ; then one can use the fact that R with the Euclidean topology is pathwise connected to show that there are two paths ρ_1 and ρ_2 such that ρ_1 is a path in the direction $\ell_{\theta 1}$ and ρ_2 is a path in the direction $\ell_{\theta 2}$, where θ_1 , θ_2 $\in \Theta$. But since the composition of two paths is again a path, it follows that R Θ R is a pathwise connected space.

To prove the converse, we assume that Θ has just one element, say θ, and let x, y be two points not in the same line ℓ_θ. Then there is no way to fined a path from x to y in the topological space R Θ R.

2.2 Corollary

R Θ R is connected iff Θ has more than one element.
Since the motion of a convergent sequence does not depend on the sequence itself only, but also depends on the structure of the underlying space. We deduce.

2.3 Theorem

A path $\rho : [0,1] \rightarrow$ R Θ R from z_1 to z_2 has image ρ ([0,1]) which is contained in a finite number of connected segments with directions being in Θ.

proof

Since $I = [0,1]$ is compact and ρ is continuous, ρ (I) is so, and consequently it is contained in a finite number of lines with slopes in Θ. Also ρ (I) is connected, it follows that $\rho (I) = \bigcup_{i=1}^{n} \ell_{\theta i}$ with the property that $I_j \bigcap I_i = \phi \; \forall \; i \neq j$ and I_{θ_i} , represents a segment in the line

ℓ_{θ_i} , $\theta_i \in \Theta$.

2.4 Definition

Let z be a point in $R\Theta R$. An open interval I_θ which lies on ℓ_θ, $\theta \in \Theta$, and contains z is called a Θ - interval at z. a Θ-star at z is defined as :

$$\Theta(z) = \bigcup_{\theta\in\Theta} I_\theta(z)$$

Obviously, $\Theta(z)$ is connected in $R\Theta R$.

2.5 Lemma [2]

A set U is $R\Theta R$ - open if and only if for each $Z \in U$, U contains Θ - star at Z.

2.6 Theorem

$R\Theta R$ is locally connected .

proof

we shall prove that the connceted open sets of $R\Theta R$ forms a basis for its topology. Let U be an open set of $R\Theta R$ and $z \in R\Theta R$. Then. by lemma (2.5) there is a Θ-star N_1 at z and $N_1 \subseteq U$. For each $z' \in N_1$, U is a nbd. of z' and consequently there exists a $\Theta(z\ z)$ such that $\Theta(z) \subseteq U$. Since $z' \in N_1 \bigcap \Theta(z)$ and since both N_1, and $\Theta(z)$ are connected, it follows that $N_2 = N_1 \bigcup (\bigcup_{z' \in N} \Theta(z))$ is connected. We continue in this way, so $N_{r+1} = N_r \bigcup$

$\bigcup_{z'\in N} \Theta(\, z'\,)\,)$ for all $r \in N$. Obviously $N_{r+1} \subset N_r$. But $N(z) = \bigcup_{r=1} N_r$. Then $N(z) \subseteq U$ and $N(z)$ is connected. Also by construction $N(z)$ contains a Θ-star for each of all its points, so by the previous lemma $N(z)$ is open.

The underlying space has some disadvantages. One of these is the following.

2.7 Theorem

If Θ has more than one element, then $R\Theta R$ is not first countable.

proof

Suppose not, then there would be a countable open base $\{ V_n, n \in N \}$ at $z \in R\Theta R$. Let us also suppose $V_n \supseteq V_{n+1}$, $n \in N$. One then can construct a Θ-sequence $Z = \langle z_n \rangle_n$, $z_n \in V_n$ such that $z_n \to z$ in $R\pi R$. Choosing $z_n \in V_n \bigcap N_{1/n}(z)$. Where $N_{1/n}(z)$ is the Euclidean open nbd. Of z of radius 1/n and $W = V_1 - Z$, one can see W is and $R\Theta R$-open set, $z \in W$. So W is a nbd. Of z and contains no z_n; hence $\{ V_n, n \in N \}$ is not a nbd. base.

From the above theorem one can conclude that $R\Theta R$ is not metrizable. There are a limited number of T_2 - space which are not T_3 - space and in our hand one of such spaces. It is the Θ-space.

3. Homology and Homotopy of $R\Theta R$

A space X is contractible to a point $x_0 \in X$ with x_0 held fixed if there is a map

$$F: X \times [0,1] \to X \quad \text{such that}$$
$$F(x,0) = x_0 \qquad x \in X$$
$$F(x,1) = x$$

$$F(x_0,t) = x_0 \qquad 0 \leq t \leq 1.$$

A space X is simply connected if X is pathwise connected and its fundamental group $\pi_1(X) \cong 0$.

It is well known that each contractible space is simply connected [1].

Let us prove that l

3.1 Lemma :

RΘR is contractible

Proof:

Define F: RΘR ×[0,1] → RΘR

$$(x,t) \rightarrow (1-t)x.$$

This is a continuous map and meets the conditionds of contractibility and the identity map on RΘR is homotopic to a constant map .

Then we conclude the fundamental of RΘR is trivil

i.e . π_1 (RΘR) ≅ 0.

It is known that if each two spaces are homotopy equivalent, then their homology grops are isomorphic . Thus

$$\mathbf{H}_0 (R\Theta R) = Z \text{ and } \mathbf{H}_n (R\Theta R) = 0 \text{ for all } n \geq 1$$

This comes by knowning the homology groups of $R\pi R$.

We there for have in our hands a counter example of non-metric space which has the same homology groups of the Euclidean space on the base set R

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مجموعات الهومولوجى و الهوموتوبى للفراغ Θ

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الملخص العربى

تم استنتاج مجموعـات الهومولوجى والهوموتوبى لهذا الفـراغ وبذلـك أصبح لدينا فراغ ليس متريا وله نفس مجموعـات الهومولوجـى والهوموتوبـى للفراغ الاقليدى بالمستوى.