APPLYING MATRIX ARITHMETIC TO FIND GEAR RATIOS

S. A. GHARRAF

Ass.Professor,Prod.Eng.Dept.Alex.University

ABSTRACT

With a properly formulated matrix equation,we can quickly locate the region in which acceptable gear train ratios lie and they can be readily programmed for either calculator or computer operations.

INTRODUCTION

Finding siutable change gears for a specified ratio is actually two problems in one:-

- Find rational number N/D close enough to the specified ratio i.
- Factor N and D into acceptable gear tooth numbers.

The first part of this problem has received a great deal of attention,while the second part still must be solved by the cut-and-try method,or by refernce to factor tables.

Matrix arithmetic offers a systematic way to determine all fractions within specified limits of a given ratio.It convergs rapidly to the region in which i lies,it is not subjected to comulative error,and it is easily arranged for desk calculator or computer work.

METHOD ANALYSIS

The method is based upon conjugate fraction theory which states that two fractions a/b and c/d are conjugate if (ab - bc) = $\pm$ 1.

This means that between two conjugate fractions there exists no other fraction with smaller numerator or denominators. In this sense each conjugate fraction is "a best approximation" of the other. If the numbers a and b are chosen so that a/b is a fraction in lowest terms greater than the desired ratio i, and if c and d are similarly chosen so that c/d is a fraction in lowest terms less than the desired ratio i, the basic matrix can be formed:

$$B = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

In matrix theory B is unimodular if (ad - bc) = 1.
A multiplier matrix is now needed and can be formed from the unit matix:

$$X = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

By adding adjacent row elements and interposing their sum:

$$\text{First stage} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\text{Second stage} = \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{vmatrix}$$

$$\text{Third stage} = \begin{vmatrix} 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 \end{vmatrix}$$

This is a new concept in matrix arithmetic which was specially developed to solve gear ratios. Note that:-

- The unit matrix can be expanded indefinitely,
- When regarded as a fraction each column is in lowest terms,
- The fractions follow a descending order,
- Each adjacent 2x2 sub-matrix is unimodular,
- No fraction with smaller terms exists with value between adjacent fractions.

These unique properties of the multiplier matrix can be used to find the specified ratio i which lies between the two ratios in the basic matrix by multiplying the two matrices:

$$B\,X = G$$

To show how this works in a practical way, suppose $i = 1/\sqrt{\pi}$ $= 0.564189$ is to be approximated within $\pm$ 0.00002 from gear ratio tables select:-

$a/b = 35/62 = 0.564516$
$c/d = 22/39 = 0.564102$

Since $35 \times 39 - 22 \times 62 = 1$, the basic matrix is unimodular, and it can be multiplied by the X matrix to give:-

$$\text{Unit matrix} = \begin{vmatrix} 35 & 22 \\ 62 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 35 & 22 \\ 62 & 39 \end{vmatrix}$$

$$\text{First stage} = \begin{vmatrix} 35 & 22 \\ 62 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 35 & 57 & 22 \\ 62 & 101 & 39 \end{vmatrix}$$

$$\text{Second stage} = \begin{vmatrix} 35 & 22 \\ 62 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 35 & 92 & 57 & 79 & 22 \\ 62 & 163 & 101 & 140 & 39 \end{vmatrix}$$

$$\text{Third stage} = \begin{vmatrix} 35 & 22 \\ 62 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 35 & 127 & 92 & 149 & 57 & 136 & 79 & 101 & 22 \\ 62 & 225 & 163 & 264 & 101 & 241 & 140 & 179 & 39 \end{vmatrix}$$

The desired ratio lies between 101/179 and 22/39 so further effort needs to be concentrated only in this area of the G matrix.

This can be done by expanding the lower portion of the X matrix to give:-

$$X = \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 7 & 4 & 5 & 1 \end{vmatrix}$$

Multiplying this with a portion of the B matrix gives:-

$$B\,X = G$$

$$\begin{vmatrix} 101 & 22 \\ 179 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 7 & 4 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 101 & 224 & 123 & 145 & 22 \\ 179 & 397 & 218 & 257 & 39 \end{vmatrix}$$

It appears now that 145/257 = 0.564202 is acceptably close to the specified i.Unfortunately,257 is a prime number and it can't be factored into suitable sets of change gears.To proceed from here the lower portion of the X matrix

$$\begin{vmatrix} 1 & 1 & 0 \\ 4 & 5 & 1 \end{vmatrix}$$

Can be expanded further and the resulting X matrix can then be multiplied:

$$\begin{vmatrix} 123 & 22 \\ 218 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 4 & 9 & 5 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 123 & 268 & 145 & 167 & 22 \\ 218 & 275 & 257 & 296 & 39 \end{vmatrix}$$

This might yield a favorable ratio.

A second alternative is to select one or two new B matrices from the above G matrix and multiply them by an expanded unit matrix to give:-

$$\begin{vmatrix} 123 & 145 \\ 218 & 257 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 123 & 391 & 268 & 413 & 145 \\ 218 & 693 & 475 & 732 & 257 \end{vmatrix}$$

or

$$\begin{vmatrix} 145 & 22 \\ 257 & 39 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 145 & 312 & 167 & 189 & 22 \\ 257 & 553 & 296 & 335 & 39 \end{vmatrix}$$

In these multiplication the unit matrix may expanded to any length necessary to produce a suitable ratio.

CONCLUSION

~~All~~ rational numbers can be formed by adding adjacent numerators and denominators starting with 1/0 and 0/1. This is the easiest way to find the desirable ratio when the tolerance is wide, but for close tolerance ratios the labor of repeated long division is excessive and the chance of error increases accordingly.

With a properly formulated matrix equation you can quickly locate the region in which acceptable ratios lie, and they can be readily programmed for either calculator or computer operation. Computationally, it is less subject to error, and, by solving a matrix equation, you identify the tolerance region in one step without involving recurrence process and continual testing of fractions. Mathematically, the properties of rational numbers is actually just a part of the larger theory of matrices.

REFERENCES

1. Merritt, H.E. " Calculation of Change Gear Combination " Journal Mechanical Engineering Science, Vol 12 No 1 1970.

NOMENCLATURE

G, B, X	............2 row unimodulator matrices
B^{-1}	.... is inverse of B matrix
i	.... gear-train ratio
N	.... numerator of i
D	.... denominator of i
a, b, c, d	.... elements of B matrix, positive integers
a/b and c/d	.... gear ratios taken from standard tables